PRELIMINARY EXAMINATION: NUMERICAL ANALYSIS II
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Work all 3 of the following 3 problems.

1. Consider the system of two ordinary differential equations
   \[ u'(t) = f(v) \quad \text{and} \quad v'(t) = g(u), \]
   where \( u(0) = u_0 \) and \( v(0) = v_0 \), and, for \( h > 0 \), the numerical scheme
   \[ U_{n+1} = U_n + h f\left( V_n + \frac{h}{2} g(U_n) \right) \quad \text{and} \quad V_{n+1} = V_n + h g\left( \frac{1}{2} (U_n + U_{n+1}) \right). \]
   \[(a)\] Show that the local truncation error for both \( u \) and \( v \) is \( O(h^2) \).
   \[(b)\] For the linear system where \( f(v) = \lambda v \) and \( g(u) = -\mu u \), both \( \lambda \) and \( \mu \) being positive, show that when \( h \lambda < 1 \) and \( h \mu < 1 \), the scheme is stable. [Hint: The eigenvalues of the matrix \( \begin{pmatrix} a & b \\ c & a \end{pmatrix} \) are \( a \pm \sqrt{bc} \).]

2. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a polygonal boundary. Consider the elliptic partial differential equation for \( u(x) \) given by
   \[ -a \Delta u + cu = f \quad \text{in} \ \Omega, \]
   \[ u = 0 \quad \text{on} \ \partial \Omega, \]
   where \( a(x) \) and \( c(x) \) satisfy \( 0 < a_* \leq a(x) \leq a^* < \infty \), \( 0 \leq c(x) \leq c^* < \infty \), and also \( |\nabla a(x)| \leq b^* < \infty \). Assume that \( f \in L^2(\Omega) \).
   \[(a)\] Find a variational form suitable for approximation by finite elements.
   \[(b)\] Give a reasonable condition on \( b^* \) that insures that your bilinear form is coercive.
   \[(c)\] Derive a bound on the error between \( u \) and a finite element approximation \( u_h \).

3. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a polygonal boundary. Consider the parabolic partial differential equation
   \[ u_t - \Delta u = f(x,t) \quad \text{for} \ x \in \Omega, \ t > 0, \]
   \[ u(x,t) = 0 \quad \text{for} \ x \in \partial \Omega, \ t > 0, \]
   \[ u(x,0) = u_0(x) \quad \text{for} \ x \in \Omega, \ t = 0. \]
   It has the variational form
   \[ (u_t, v) + (\nabla u, \nabla v) = (f, v) \quad \forall v \in H^1_0(\Omega). \]
   \[(a)\] Write down the discrete scheme that uses a suitable finite element method in space and backward Euler in time.
   \[(b)\] Show that your scheme is stable by bounding
   \[ \max_n \|u^n\|^2 + \sum_n \|
\nabla u^n\|^2 \Delta t \]
   in terms of \( \|u_0\| \) and \( \max_t \|f(\cdot, t)\| \).