The Preliminary Examination in Probability
Part I

Aug 2011

Problem 1 (36pts). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and let \(\{X_n\}_{n \in \mathbb{N}}\) be a sequence of random variables bounded in \(L^1\). We say that \(\{X_n\}_{n \in \mathbb{N}}\) converges to a random variable \(X \in L^1\) in the biting sense if, for each \(\varepsilon > 0\), there exists \(A_\varepsilon \in \mathcal{F}\) such that \(\mathbb{P}[A_\varepsilon] > 1 - \varepsilon\) and \(X_n1_{A_\varepsilon} \to X1_{A_\varepsilon}\) in \(L^1\).

1. Show that the biting limit \(X\) of the sequence \(\{X_n\}_{n \in \mathbb{N}}\) is a.s.-unique (provided it exists).
2. Show that convergence in \(L^1\) implies the biting convergence, but that the biting convergence does not necessarily imply convergence in \(L^1\).
3. Show that the two concepts coincide for uniformly integrable sequences \(\{X_n\}_{n \in \mathbb{N}}\).

Problem 2 (32pts). Let \(\{X_n\}_{n \in \mathbb{N}}\) be a sequence of random variables such that \(\mathbb{P}[X_n \in \mathbb{N}_0] = 1\), for all \(n \in \mathbb{N}\); here \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\). Show that for a random variable \(X\) with \(\mathbb{P}[X \in \mathbb{N}_0] = 1\), the following are equivalent:

1. \(X_n \to X\) in distribution,
2. \(\mathbb{P}[X_n = k] \to \mathbb{P}[X = k]\), for all \(k \in \mathbb{N}_0\),
3. \(g_{X_n}(t) \to g_X(t)\) for all \(t \in [0, 1]\), where, for any \(Y\), we define \(g_Y(t) = \sum_{k \in \mathbb{N}_0} \mathbb{P}[Y = k] t^k\).

(Hint: For (3) \(\Rightarrow\) (1), use the ideas from the proof of the Continuity Theorem for weak convergence. To deal with tightness, show that \(\mathbb{P}[Y \geq N] \leq 2 \varepsilon \int_0^\infty (1 - g_Y(t))\,dt\), for any random variable \(Y\) with \(\mathbb{P}[Y \in \mathbb{N}_0] = 1\) and any \(N \in \mathbb{N}\), \(\varepsilon > 0\) with \(\varepsilon^N \leq 1/2\).)

Problem 3 (32pts). Let \(\{\varepsilon_n\}_{n \in \mathbb{N}_0}\) be an iid sequence with \(\mathbb{P}[\varepsilon_n = 1] = 1 - \mathbb{P}[\varepsilon_n = -1] = p \in (1/2, 1)\). We interpret \(\{\varepsilon_n\}_{n \in \mathbb{N}_0}\) as outcomes of a series of gambles. A gambler starts with \(Z_0 > 0\) dollars, and in each play wagers a certain portion of her wealth. More precisely, the wealth of the gambler at time \(n \in \mathbb{N}\) is given by

\[
Z_n = Z_0 + \sum_{k=1}^{n} C_k \varepsilon_k,
\]

where \(\{C_n\}_{n \in \mathbb{N}_0}\) is a predictable process such that \(C_k \in [0, Z_{k-1}]\), for \(k \in \mathbb{N}\). The goal of the gambler is to maximize the “return” on her wealth, i.e., to choose a strategy \(\{C_n\}_{n \in \mathbb{N}_0}\) such that the expectation \(\frac{1}{p} \mathbb{E}[\log(Z_T/Z_0)]\), where \(T \in \mathbb{N}\) is some fixed time horizon, is the maximal possible.

1. Define \(\alpha = H(\frac{1}{2}) - H(p)\), where \(H(p) = -p \log p - (1-p) \log(1-p)\) is the entropy function, and show that the process \(\{W_n\}_{n \in \mathbb{N}_0}\) given by

\[
W_n = \log(Z_n) - \alpha n, \quad \text{for } n \in \mathbb{N}_0
\]

is a supermartingale. Conclude that \(\mathbb{E}[\log(Z_T)] \leq \log(Z_0) + \alpha T\), for any choice of \(\{C_n\}_{n \in \mathbb{N}_0}\).
2. Show that the upper bound above is attained for some strategy \(\{C_n\}_{n \in \mathbb{N}_0}\).