Part I

Do three of the following five problems.

1. Let $p$ be a prime.
   1a. Classify elements of the symmetric group $S_p$ of order $p$.
   1b. Let $G$ be a finite group and $H$ be a subgroup of index $p$. Assume $p$ is the smallest prime number that divides the order of $G$. Show that $H$ is normal. [Hint: if $H$ is not normal, how many conjugates will it have?]

2a. Show that any matrix in $\text{GL}_n(\mathbb{C})$ of finite order is diagonalizable.
   2b. Let $G \subset \text{GL}_n(\mathbb{C})$ be a finite abelian subgroup of the invertible $n$ by $n$ complex matrices. Show that $G$ is conjugate to a subgroup of $\text{GL}_n(\mathbb{C})$ whose elements are all diagonal matrices.

3. Show that, up to isomorphism, there are exactly four groups of order 170.

4a. Let $A$ be an $n$ by $n$ matrix with entries in a field $F$, and suppose that the minimal polynomial of $A$ is equal to the characteristic polynomial of $A$. Show that for any $n$ by $n$ matrix $B$ with entries in $F$ such that $BA = AB$, there exists a polynomial $P \in F[t]$ such that $B = P(A)$.
   4b. Show that if the characteristic polynomial of $A$ is not equal to the minimal polynomial of $A$, then there exists an $n$ by $n$ matrix $B$ with entries in $F$ such that $BA = AB$ but $B$ is not equal to $P(A)$ for any polynomial $P$.

5. Let $G$ be a $p$-group, and let $H$ be a normal subgroup of $G$. Show that $H \cap Z(G)$ contains an element other than the identity.
Algebra Part II

1. Let $F$ be a finite field of odd characteristic. Prove that the product of the nonzero elements of $F$ is equal to $-1$.

2. Let $E$ be the splitting field of the polynomial $x^5 - 3$ over $\mathbb{Q}$.
   2a. Determine the Galois group $\text{Gal}(E/\mathbb{Q})$ as a group of permutations of the roots of $x^5 - 3$.
   2b. Prove that $E$ is not a subfield of any cyclotomic extension of $\mathbb{Q}$.

3. Consider $f(x) = x^4 + 7x + 7$.
   3a. Determine the degree of the splitting field of $f(x)$ over $\mathbb{F}_3$.
   3b. Let $F$ be the splitting field of $f(x)$ over $\mathbb{Q}$. Prove that $\text{Gal}(F/\mathbb{Q}) \cong S_4$.

4. Let $q$ be a prime power. Determine, in terms of $q$, the number of irreducible monic polynomials in $\mathbb{F}_q[t]$ of degree 6.

5. Let $f \in \mathbb{Q}[t]$ be an irreducible polynomial of degree 5, and let $K$ be the splitting field of $f$ over $\mathbb{Q}$.
   5a. Suppose $f$ has exactly three real roots. Show that $\text{Gal}(K/\mathbb{Q})$ is isomorphic to $S_5$.
   5b. Now suppose $f$ has exactly one real root. What can you say about $\text{Gal}(K/\mathbb{Q})$?