(1) Let $E \subset \mathbb{R}$ be a measurable set such that $0 < |E| < \infty$. Prove that for every $\alpha \in (0, 1)$ there is an open interval $I$ such that $|E \cap I| \geq \alpha |I|$. 

(2) Let $Z$ be a subset of $\mathbb{R}$ with measure zero. Show that the set $A = \{x^2 \mid x \in Z\}$ also has measure zero.

(3) Let $f_k \to f$ a.e. on $\mathbb{R}$. Show that given $\varepsilon > 0$, there exists $E$, with $|E| < \varepsilon$, so that $f_k \to f$ uniformly on $I \setminus E$, for any finite interval $I$.

(4) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $f \in L^1(\Omega)$. Prove that

$$\lim_{p \to 0} \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} = \exp \left( \int_{\Omega} \log |f| \, d\mu \right),$$

where $\exp[-\infty] = 0$. To simplify the problem, you may assume $\log |f| \in L^1(\Omega)$.

(5) Let $h$ be a bounded, measurable function, such that, for any interval $I$

$$\left| \int_I h \right| \leq |I|^{1/2}.$$ 

Let $h_\varepsilon(x) = h(x^\varepsilon)$. Show that for any $A$ with $|A| < \infty$

$$\int_A h_\varepsilon(x) \, dx \to 0, \text{ as } \varepsilon \to 0.$$