1. Let \( f \in C^\infty(\mathbb{R}) \), bounded with compact support, and
\[
g(x) := f(x) + \frac{1}{10} \sin(10\pi x).
\]

(a) Find an even polynomial \( K(x) \) satisfying
\[
\int_{-1}^{1} x^k K(x) dx = \begin{cases} 
1, & k = 0, \\
0, & k = 1, 2, 3,
\end{cases}
\]
and \( K(\pm 1) = 0 \).

(b) For \( \epsilon > 0 \), define
\[
K_\epsilon(x) := \begin{cases} 
\frac{1}{\epsilon} K(\frac{x}{\epsilon}), & -\epsilon \leq x \leq \epsilon, \\
0, & \text{otherwise}.
\end{cases}
\]

Show that there is a constant \( C \) such that for sufficiently small \( \epsilon \),
\[
|f(x) - K_\epsilon * g(x)| \leq C \epsilon^4.
\]

2. Let \( \theta \in (0, 1) \).

(a) Determine \( \alpha, \beta, \) and \( \gamma \) such that the quadrature \( \alpha f(1) + \beta f(\theta) + \gamma f(0) \) yields the exact value of
\[
\int_{0}^{1} f(x) dx
\]
for all quadratic polynomials \( f(x) \).

(b) Define
\[
f_\theta(x) = \begin{cases} 
1, & \text{if } x \leq \theta^2, \\
0, & \text{otherwise}.
\end{cases}
\]

Show that with the above choice of \( \alpha, \beta, \) and \( \gamma \),
\[
\lim_{\theta \to 0^+} \left| \int_{0}^{1} f_\theta(x) dx - (\alpha f_\theta(1) + \beta f_\theta(\theta) + \gamma f_\theta(0)) \right| = \infty.
\]
3. Let $A$ be a real, positive definite, self-adjoint matrix. Define the energy

$$F(y) := \frac{1}{2}(x - y)^T A(x - y).$$

Consider the iterative scheme

$$x^{n+1} = x^n - s_n r_n,$$

where

$$s_n := \frac{||r_n||^2}{r_n^T A r_n}, r_n := Ax^n - b. \quad (1)$$

(a) Show that given $x^n$, the choice of $s^n$ in (1) minimizes

$$E(x^{n+1}) := \frac{1}{2}(x^{n+1})^T A x^{n+1} - (x^{n+1})^T b.$$

(b) Show that $F(x^n)$ tends to 0 as $n$ tends to $\infty$, and therefore $x^n$ converges to the solution of $Ax = b$. 