1. Consider the energy

\[ E(u) := \frac{1}{2} \int_0^1 |u_x(x)|^2 \, dx + \frac{\lambda}{2} \int_0^1 (u(x) - f(x))^2 \, dx, \lambda > 0, \]

defined for \( u \in C^2([0,1]; \mathbb{R}) \), \( f \in C([0,1]; \mathbb{R}) \), and \( u(0) = u(1) = f(0) = f(1) = 0 \). Consider a discrete approximation of \( E \) as follows: \( U = (u_1, u_2, \ldots, u_{N-1})^T \),

\[ E_h(U) := \frac{1}{2} \sum_{j=1}^{N-1} |D^+ u_j|^2 h + \frac{\lambda}{2} \sum_{j=1}^{N-1} |u_j - f_j|^2 h, \]

where \( h = 1/N \), \( u_0 = u_N = 0 \), \( D^+ u_j = (u_{j+1} - u_j)/h \), and \( f_j = f(jh) \).

(a) Derive the linear system

\[ AU = b \tag{1} \]

whose solution minimizes \( E_h \).

(b) Derive the Gauss-Seidel method for this linear system and show that the iteration method will converge.

(c) Does the solution of (1) approximate the minimizer of \( E \)? Justify your answer.

2. Consider

\[ u_t = a(x)u_x, \quad 0 < x < 1, \ t > 0. \]

(a) Derive an upwind scheme for the equation. Determine a suitable boundary condition such that the PDE is well-posed. Introduce a suitable discrete \( L^2 \) norm \( \| \cdot \|_h \) and show that the upwind scheme is stable in this norm.

(b) Derive a discontinuous Galerkin method with piecewise linear basis functions for the equation above.

3. Derive the order of accuracy of the multistep method

\[ y_{n+1} + 4y_{n+1} - 5y_n = h(4f_{n+1} + 2f_n), \ h > 0 \]
for approximation of the solutions of $y' = y$. Is the scheme convergent? Why?