Preliminary Examination in Algebra–Fall semester
August 17, 2012, RLM 9.166, 1:00-2:30 p.m.

Do three of the following four problems.

1. Let $G$ be a group, $\text{Aut}(G)$ the group of all automorphisms $\varphi : G \to G$, and

   $$Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$$

   the center of $G$.

   (i.) For each $h$ in $G$ define a map $\psi_h : G \to G$ by

   $$\psi_h(g) = hgh^{-1}.$$ 

   Show that each map $\psi_h$ is an element of $\text{Aut}(G)$, and the set $\text{Inn}(G) \subseteq \text{Aut}(G)$ of all such automorphisms is a normal subgroup of $\text{Aut}(G)$.

   (ii.) Prove that $\text{Inn}(G)$ is isomorphic to the quotient group $G/Z(G)$.

   (iii.) Assume that $G$ is a finite abelian group such that the number $|\text{Aut}(G)|$ of elements in $\text{Aut}(G)$ is odd. Prove that $|G|$ is either 1 or 2.

2. Let $n$ be a square free integer greater than 3, and let $R$ be the ring $\mathbb{Z}[x]/(x^2 + n)$, where $(x^2 + n)$ is the principal ideal generated by $x^2 + n$.

   (i.) Show that each of the elements $2$, $x$, and $1 + x$, is irreducible in $R$.

   (ii.) Show that $R$ is not a unique factorization domain.

   (iii.) Give an example of an ideal in $R$ that is not principal.

3. A commutative ring $A$ is called Artinian if every descending chain of ideals

   $$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

   is eventually constant.

   (i.) Show that an Artinian integral domain is a field.

   (ii.) Show that in an Artinian ring $A$, every prime ideal is maximal.

4. Let $K$ be an algebraically closed field and $K^N$ the $K$-vector space of $N \times 1$ column vectors with entries in $K$. Let $A$ be an $N \times N$ matrix with entries in $K$. We say that a vector $\mathbf{v}$ in $K^N$ is a cyclic vector for $A$ if the set

   $$\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, A^3\mathbf{v}, \ldots, A^{N-1}\mathbf{v}\}$$

   is a basis for $K^N$. Prove that $A$ has a cyclic vector if and only if the characteristic polynomial for $A$ and the minimal polynomial for $A$ are equal.