Numerical Analysis Exam: Part B, August 2012

1. The following ordinary differential equation and corresponding predictor-corrector (P-C) method are given,

\[
y' = f(y), \quad t > 0; \quad y(0) = y_0
\]

\[
y^{*}_{n+1} = y_n + hf(y_n), \quad n = 1, 2, \ldots
\]

\[
y_{n+1} = y_n + h\left(f(y_n) + f(y^{*}_{n+1})\right)/2
\]

\[
\left( y_n = y(t_n), \quad t_n = nh, \quad n = 1, 2, \ldots \right)
\]

(a) Determine the order of the approximation and whether the method is stable.
(b) Determine the region of absolute stability.
(c) Describe how step size control can be based on this P-C technique.

2. Consider the elliptic partial differential equation,

\[-(a(x,y)u_x)_x - (a(x,y)u_y)_y + b(x,y)u = f(u), \quad (x,y) \in \Omega
\]

\[a(x,y) \geq a > 0, \quad b(x,y) \geq b > 0, \quad \Omega = \{(x,y), 0 < x < 1, 0 < y < 1\}
\]

Boundary \(\partial \Omega\): \(u = 0\) for \(y = 0, 1\), 0 \(\leq x \leq 1\).

\[u_x = cau \quad \text{for } x = 0, 1, \quad 0 < y < 1
\]

(a) Reformulate this boundary value problem on weak form and describe a finite element approximation based on triangulation of \(\Omega\) and piecewise polynomials.
(b) Prove that the bilinear and linear forms related to the weak formulation satisfies the standard conditions for convergence when \(\alpha = 0, f(u) \equiv 0\).
(c) Present solution methods for the nonlinear algebraic system resulting from the discretization in part (a) above.

3. Approximate the partial differential equation below,

\[u_t + Au_x = f(x,t), \quad 0 < x < 1, t > 0,
\]

\[u(x,0) = u_0(x), \quad 0 \leq x < 1, \quad \text{periodic boundary condition}
\]

(a) Define the standard upwind approximation of the equation when \(A\) is a real number and determine its order of approximation.
(b) Prove that this scheme is stable and satisfies a maximum principle under suitable condition on \(\Delta t/\Delta x\) (the step sizes) when \(A > 0, \alpha = 0, f(x,y) \equiv 0\).
(c) Use von Neumann analysis to prove stability in \(L^2\) when \(A\) is a symmetric matrix and the upwind scheme is replaced by Lax Friedrich. Express the stability condition (CFL condition) in terms of the spectral radius of \(A\).