Preliminary Examination in Topology: August 2012
Algebraic Topology portion

Instructions: Do all three questions.
Time Limit: 90 minutes.

1. Let $R_1$ be a rectangle and identify opposite edges to give a torus $T$. Let $R_2$ be a rectangle and identify opposite edges to give a Klein bottle $K$. Let $S$ be the space gotten by further identifying the $a$-edge of $R_1$ with the $a$-edge of $R_2$, and the $b$-edge of $R_1$ with the $b$-edge of $R_2$ as indicated in the figure below.

![Diagram of rectangles and edges labeled a and b]

(1) Use a Mayer-Vietoris sequence to compute the first and second homology groups of $S$. You may take the homology groups of 1 and 2-manifolds as facts.
(2) Is there a deformation retraction of $S$ onto $K$?
(3) Is $T$ a retract of $S$?

2. Thinking of $S^1$ as the unit circle in $R^2$, let $f: S^1 \to S^1$ be such that $f(-x) = -f(x)$.

(1) Let $C$ be the quotient of $S^1$ under the equivalence relation $x \sim -x$ for $x \in S^1$ and let $p: S^1 \to C$ be the quotient map. Then $f$ induces a map $g: C \to C$ so that $p \circ f = g \circ p$. Show that $g$ induces an injection on fundamental groups.
(2) Show that $f$ is not homotopic to a constant map.
(3) Consider $S^3$ as the unit sphere in $R^4$. Use (2) to show that there is no map $h: S^3 \to S^1$ such that $h(-x) = -h(x)$.

3. (1) What are the spaces that 2-fold cover the Mobius band?
(2) Let $X$ be the connected sum of three projective planes. Show that there is an orientable surface $Y$ that 2-fold covers $X$ by drawing a picture of the covering space and covering map. Show that, up to equivalence of covers, there is a unique 2-fold covering of $X$ by an orientable surface. (Hint: Show that the covering corresponds to the kernel of a unique map on fundamental group, by picking the right generators of $\pi_1(X)$).