ALGEBRA PRELIMINARY EXAM: PART I

Problem 1

Fix an \( n \times n \) matrix \( A \) with entries in an algebraically closed field \( K \). Let \( V \) be the space of \( n \times n \) matrices over \( K \) that commute with \( A \). Observe that \( V \) is a vector space over \( K \). Show that

\[ \dim V \geq n, \]

and the equality holds if and only if the characteristic polynomial of \( A \) equals the minimal polynomial of \( A \).

Problem 2

A finite group \( G \) is supersolvable if there is an increasing chain of subgroups

\[ \{1_G\} = G_0 \subset G_1 \subset \ldots \subset G_r = G \]

such that each \( G_i \) is normal in \( G \), and \( G_{i+1}/G_i \) is cyclic for all \( i \).

(a) Show that every \( p \)-group is supersolvable.
(b) Give an example of a solvable group that is not supersolvable.

Problem 3

Let \( A \) be the ring of \( n \times n \) matrices over a field \( F \).

(a) Show that for any subspace \( V \) of \( F^n \), the set \( I_V \) of matrices whose kernel contains \( V \) is a left ideal of \( A \).
(b) Show that every left ideal of \( A \) is principal.

Date: August 22, 2013.