ALGEBRA PRELIMINARY EXAM: PART II

Choose two of the following three problems.

PROBLEM 1

Let $K/F$ be a finite Galois extension and $g \in \text{Gal}(K/F)$. Compute the characteristic polynomial of $g$, where $g$ is considered as an $F$-linear map from $K$ to $K$.
(Hint: first consider the case where $K/F$ is cyclic, i.e., $\text{Gal}(K/F)$ is a cyclic group.)

PROBLEM 2

Consider the polynomial $f(x) = x^6 - 4x^3 + 1$. Let $L$ be the splitting field of $f(x)$ over $\mathbb{Q}$.
(a) Show that
(i) $f(x)$ has two real roots: $\alpha$ and $\alpha^{-1}$.
(ii) $x^3 - 1$ splits in $L$.
Let $\zeta \in L$ be a primitive cube root of unity.
(b) Determine the degree of $L/\mathbb{Q}$. (You may use without proof the fact that when viewed modulo 5 the polynomial $f(x)$ does not have any quadratic factors in $\mathbb{F}_5[x]$.)
(c) Prove that $\sqrt[3]{5} \notin L$.
(d) Prove that $\text{Gal}(L/\mathbb{Q}) \cong D_{12}$, here $D_{12}$ denotes the dihedral group of order 12.
(Hint: consider the action of $\text{Gal}(L/\mathbb{Q})$ on the roots of $f(x)$.)
(e) Use $\alpha$ and $\zeta$ to describe all the subfields $F \subseteq L$ such that $L/F$ is quadratic and $L/\mathbb{Q}$ is Galois.

PROBLEM 3

Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial and $p \in \mathbb{Z}$ be a prime. Consider the reduction of $f(x)$ modulo $p$, denoted $\overline{f}(x) \in \mathbb{F}_p[x]$, and assume that $\overline{f}$ has no multiple roots.
Let $L/\mathbb{Q}$ be the splitting field of $f$. Consider the roots of $f(x)$
$$\alpha_1, \ldots, \alpha_r \in L,$$
and the subring of $L$ that they generate
$$A := \mathbb{Z}[\alpha_1, \ldots, \alpha_r] \subseteq L.$$ You may use without proof the fact that $A \cap \mathbb{Q} = \mathbb{Z}$.
Set $G := \text{Gal}(L/\mathbb{Q})$ and $\overline{G}$ to be the Galois group of $\overline{f}$ over $\mathbb{F}_p$.

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(a) Consider the set $S_p$ of maximal ideals $Q \subseteq A$ such that $Q \cap \mathbb{Z} = p\mathbb{Z}$. Show that the set $S_p$ is non-empty, and that the action of $G$ on $L$ induces an action of $G$ on $S_p$.

(b) Fix $P \in S_p$. Let $H \subseteq G$ be the stabilizer of the ideal $P$ in $G$.
   (i) Show that the choice of the maximal ideal $P \in S_p$ induces a homomorphism $\pi : H \to \overline{G}$.
   (ii) Prove that the homomorphism $\pi : H \to \overline{G}$ is injective.
       (Hint: consider the action of $\pi(h)$ on the roots of $\overline{f}$ for $h \in H \setminus \{1_G\}$.)

(c) For every $a \in A$, there exists $t_a \in A$ such that
   $$t_a \equiv a \mod P,$$
   $$g(t_a) \in P \text{ for } g \notin H.$$  

The existence of $t_a$ (which you may assume without proof) is a consequence of the Chinese Remainder Theorem for the ring $A$. Using $t_a \in A$ as above, we define the polynomial
   $$w_a(x) = \prod_{g \in G} (x - g(t_a)) \in A[x].$$

(i) Show that $w_a(x) \in \mathbb{Z}[x]$, and let $\overline{w}_a(x)$ be its reduction modulo $p$. Show that if the reduction $\overline{a}$ (mod $P$) is non-zero, then every conjugate of $\overline{a}$ is of form $\overline{h}(\overline{a})$ for some $h \in H$.
   (ii) Prove that $\pi : H \to \overline{G}$ is an isomorphism.

(d) Consider the factorization of $\overline{f}$ into irreducible factors in $\mathbb{F}_p[X]$  
   $$\overline{f} = \overline{g}_1 \cdot \overline{g}_2 \cdots \overline{g}_r$$
   where $d_i = \deg(\overline{g}_i)$. Prove that there exists an element $h \in H$ of cycle type $(d_1, d_2, \ldots, d_r)$, here $h$ is viewed as a permutation of the roots of $f$.

Recall the cycle type refers to the lengths of the cycles when you express a permutation as a product of disjoint cycles. E.g. the permutation $(12)(345) \in S_5$ has cycle type $(2, 3)$. 