PROBLEM 1

(a) Provide an example of a sequence of measurable functions on $[0,1]$ which converges in $L^1$ to the zero function but does not converge pointwise a.e.

(b) Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of integrable functions on $[0,1]$ such that $\|f_n\|_{L^1} \leq n^{-2}$ holds for all $n$. Show that $\{f_n\}_{n=1}^\infty$ converges pointwise a.e. to the zero function.

PROBLEM 2

Let $(x_1, x_2, \ldots)$ be an arbitrary sequence of real numbers in $[0,1]$ (possibly dense). Show that the series $\sum_k k^{-3/2}|x - x_k|^{-1/2}$ converges for almost every $x \in [0,1]$.

PROBLEM 3

Assume that $\mu$ is a finite Borel measure on $\mathbb{R}^n$, and that there exists a constant $0 < R < \infty$ such that the $k$-th moments of $\mu$ satisfy the bound

$$\int d\mu(x)|x|^k < R^k \quad \forall k \in \mathbb{N},$$

for some $0 < r \leq 1$. Prove that $\mu$ has bounded support contained in $\{x \in \mathbb{R}^n : |x| \leq R\}$ if $r = 1$, and in $\{x \in \mathbb{R}^n : |x| \leq 1\}$ if $0 < r < 1$.

PROBLEM 4

Let $f$ be a continuous function on $[0,1]$. Find

$$\lim_{n \to \infty} n \int_0^1 x^n f(x) \, dx.$$

Justify your answer.