Problem 2.1. Let $X_n \geq 0$ be a sequence of random variables such that there exist $0 < \alpha < \beta$ with the property that

$$
\mathbb{E}[X_n^\alpha] \to 1, \quad \mathbb{E}[X_n^\beta] \to 1
$$

for $n \to \infty$. Show that $X_n \to 1$ in probability.

Problem 2.2. Let $X_0 = (1, 0)$ and define $X_n \in \mathbb{R}^2$ inductively by declaring that $X_{n+1}$ is chosen at random from the ball of radius $|X_n|$ centered at the origin, i.e., $X_{n+1}/|X_n|$ is uniformly distributed on the ball of radius one and independent of $X_1, \ldots, X_n$. Prove that

$$
\frac{\log(|X_n|)}{n} \to c, \quad a.s.
$$

and compute $c$.

Problem 2.3. (1) Consider an adapted process $(X_n)_n$ such that for any bounded stopping time $T$ we have $X_T \in L^1$ and

$$
\mathbb{E}[X_T] = \mathbb{E}[X_0].
$$

Show that $X$ is a martingale.

(2) Let $(X_n)_n$ a simple but non-symmetric random walk with probabilities to go up and down $p$ and $q$. Fix a time horizon $N$ (deterministic). Compute

$$
\sup_{0 \leq T \leq N} \mathbb{E}[|X_T|]
$$

where supremum is taken over all possible stopping times.