PRELIMINARY EXAM: ALGEBRAIC TOPOLOGY

**Date:** August 2013.

**Instructions:** Do all three problems.

**Time Limit:** 90 minutes.

**Problem 1.** Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and let $s_0$ be the point $(1,0,\ldots,0) \in S^n$. Define $\pi_n(X,x_0)$ to be the set of homotopy classes of maps $f : (S^n,s_0) \to (X,x_0)$. Let $p : (\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a covering projection. Show that for $n \geq 2$ the map $p_* : \pi_n(\tilde{X},\tilde{x}_0) \to \pi_n(X,x_0)$ defined by $p_*([f]) = [p \circ f]$ is a bijection.

**Problem 2.** Let $T$ be a 2-torus, and $T_0$ a 2-torus with the interior of a disk removed. Let $C \subset T$ be the simple closed curve illustrated below and $C_0 \subset T_0$ be the boundary of the removed disk. Let $X$ be the space obtained by gluing $T$ and $T_0$ together via a homeomorphism $h : C \to C_0$.

(a) Compute the homology groups $H_q(X)$.

(b) $X$ has a 2-fold covering space $\tilde{X}$ with $\dim H_1(\tilde{X},\mathbb{Z}_2) = 5$. What is $\dim H_2(\tilde{X},\mathbb{Z}_2)$?

**Problem 3.**

(a) Define the *degree* $\deg(f)$ of a map $f : S^n \to S^n$.

(b) Let $f,g : S^n \to S^n$ be maps such that $|\deg(f)| \neq |\deg(g)|$. Show that there exists $x \in S^n$ such that $f(x)$ and $g(x)$ are orthogonal in $\mathbb{R}^{n+1}$.