PRELIMINARY EXAM: DIFFERENTIAL TOPOLOGY

Date: August 2013

Instructions: Do all three problems.

Time Limit: 90 minutes.

Problem 1. Given disjoint oriented manifolds $M, N \subset \mathbb{R}^{k+1}$, the linking map $\lambda : M \times N \to S^k$ is given by

$$\lambda(x, y) = \frac{x - y}{|x - y|}.$$ 

If $\dim(M) + \dim(N) = k$, the linking number $l(M, N)$ of $M$ and $N$ is defined to be the degree of the linking map. Show that if $M$ bounds an oriented manifold with boundary $W \subset \mathbb{R}^{k+1}$ which is disjoint from $N$, then $l(M, N) = 0$.

Problem 2. Assume that $n \geq 1$.

(a) Prove or disprove: Every smooth compact manifold $M^n$ admits a smooth vector field $v$ with only finitely many zeros. (Hint: you may assume the manifold embeds in $\mathbb{R}^N$ for some $N$.)

(b) Prove or disprove: Every smooth compact manifold $M^n$ admits a smooth vector field $v$ with no zeros.

(c) Recall that a Lie group is a smooth manifold $G$ endowed with a group structure such that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are both smooth. If $G$ is a compact Lie group, compute its Euler characteristic $\chi(G)$ and prove that your answer is correct.
Problem 3.
(a) Let \( v, w \) be vectors in \( \mathbb{R}^3 \), and let \( v', w' \) be the (orthogonal) projections of those vectors onto the plane \( P \) given by the equation \( x + 2y + 2z = 0 \). Write down a 2-covector \( \alpha_0 \) such that \( \alpha_0(v, w) \) is the (signed) area of the parallelogram spanned by \( v' \) and \( w' \). (Your answer should depend on your choice of orientation for the plane. State your choices clearly.)
(b) Write down a 2-form \( \alpha \) on \( \mathbb{R}^3 \) such that the integral of \( \alpha \) over a compact, oriented surface-with-boundary \( S \) gives the signed area of the orthogonal projection of \( S \) onto \( P \).
(c) Compute the integral of \( \alpha \) over the northern hemisphere of the unit sphere, oriented via the outward normal.