Problem 2.1. Let \((M_t)_{0 \leq t \leq T}\) be a submartingale and let \(\lambda > 0\). Show that
\[
\lambda P(\max_{0 \leq t \leq T} M_t \geq \lambda) \leq E[M_T 1_{\{\max_{0 \leq t \leq T} M_t \geq \lambda\}}].
\]
(1) Consider a one-dimensional Brownian motion \(B\) starting at \(B_0 = 0\). Let \(u, v : [0, \infty) \to \mathbb{R}\) such that \(u\) is \(C^1\), strictly increasing and \(u(0) = 0\). Assume also that \(v(t) \neq 0\) for each \(t\) and \(v\) has bounded variation. Show that the process
\[
X_t = v(t) B_u(t)
\]
is a semi-martingale (in its own filtration), and the martingale part is
\[
\int_0^t v(s) dB_u(s).
\]
(2) Show that the martingale part is a Brownian motion if and only if \(v^2(s) u'(s) = 1\) for each \(s\).
(3) Find \(u, v\) such that \(X\) defined above is an Ornstein-Uhlenbeck process with parameter \(\beta\), i.e
\[
dX_t = \beta X_t + d\gamma_t
\]
for some Brownian motion \(\gamma\).

Problem 2.2. (The range of Brownian Motion) Let \(B\) be a one-dimensional BM starting at zero. Define
\[
S_t = \max_{s \leq t} B_s, \quad I_t = \inf_{s \leq t} B_s, \quad \theta_c = \inf\{t : S_t - I_t = c\},
\]
for some \(c > 0\).
(1) Show that, for each \(\lambda\), the process
\[
M_t = \cosh(\lambda(S_t - B_t)) \exp(-\frac{\lambda^2 t}{2})
\]
is a martingale.
(2) Prove that
\[
E[\exp(-\frac{\lambda^2 \theta_c}{2})] = \frac{2}{1 + \cosh(\lambda c)}.
\]