Problem 1

Let $p$ be a prime number. Show that the $p$-Sylow subgroup of the symmetric group $S_{np}$ is an abelian group of order $p^n$ for $n < p$ and is a non-abelian group of order $p^{n+1}$ when $n = p$.

Problem 2

Prove that every prime ideal in $\mathbb{Z}[\sqrt{-5}]$ is maximal.

Problem 3

(1) Let $G$ be a finite group which acts transitively on a set $X$. For any $x \in X$ let $\text{Stab}_G(x) = \{g \in G : gx = x\}$. Prove that

$$G = \bigcup_{x \in X} \text{Stab}_G(x)$$

if and only if $X = \{x\}$ is a singleton and $gx = x$ for all $g \in G$, i.e. the action is trivial.

(2) Give an example of (an infinite) group $G$ where the above fails.

Hint: Every matrix is conjugate to an upper triangular matrix over $\mathbb{C}$.

Problem 4

Let $k$ be a field, $n$ a positive integer, and $T$ the linear transformation on $k^n$ defined by

$$T(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}).$$

We view $k^n$ as a $k[x]$-module with $x$ acting as $T$.

(1) Show that the $k[x]$-module $k^n$ is isomorphic to $k[x]/(x^n - 1)$.

(2) Let $V$ be a linear subspace of $k^n$ satisfying $T(V) \subset V$. Prove that there exists a monic polynomial $g(x) \in k[x]$ dividing $x^n - 1$ such that $V$ corresponds to

$$\{g(x)a(x) : a(x) \in k[x], \deg a(x) < n - \deg g(x)\}$$

under the above isomorphism.

(3) Take $k = \mathbb{R}$, the real numbers, and $n = 3$. Describe explicitly all subspaces $V$ of $\mathbb{R}^3$ satisfying $T(V) \subset V$.

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