1. Let $X$ and $Y$ be Banach spaces, $T \in B(X,Y)$, and $T$ be bounded below.

(a) Show that $T$ is a injective.

(b) Show that the range of $T$, $R(T)$, is closed in $Y$.

(c) Give a simple example of $T$ that is bounded below but not surjective.

(d) Define $\tilde{T}: X \to \mathbb{R}(T)$ by $\tilde{T}x = Tx$ for all $x \in X$. Show that $\tilde{T}$ is a bijective, bounded linear map.

2. Given an open set $\Omega \subset \mathbb{R}^n$ and a measurable function $a: \Omega \to \mathbb{R}$ define

$$(Tu)(x) = a(x)u(x) \quad \forall x \in \Omega.$$ 

Assume $Tu \in L^q(\Omega)$ for every $u \in L^p(\Omega)$ for some $1 \leq q \leq p \leq \infty$.

(a) Show that the map $T: L^p(\Omega) \to L^q(\Omega)$ is bounded. [Hint: Consider uniform boundedness of a sequence of approximating operators.]

(b) Show that $a \in L^r(\Omega)$, where $r = pq/(p-q)$ if $p < \infty$ and $r = q$ if $p = \infty$.

3. Let $X$ and $Y$ be Banach spaces, let $A: X \to Y$ be bounded, linear and surjective, let $B: X \to Y$ be bounded and linear, and let $\alpha = ||A-B||$.

(a) Show that there exists $\sigma > 0$ such that $\bar{B}_Y \subset A\bar{B}_X/\sigma$ for all $r > 0$, where $\bar{B}_X$ and $\bar{B}_Y$ are closed balls of radius $r$ about the origin in $X$ and $Y$, respectively.

(b) For given $f \in Y$, let $y_n \in Y$ and $x_n \in X$ be sequences such that

$$y_0 = f, \quad Ax_n = y_n, \quad \text{and} \quad y_{n+1} = y_n - Bx_n \quad \text{for } n \geq 0.$$ 

Show that the required $x_n$ can be chosen such that

$$||y_n|| \leq (\alpha/\sigma)^n||f|| \quad \text{and} \quad ||x_n|| \leq \sigma^{-1}(\alpha/\sigma)^n||f|| \quad \text{for } n \geq 0.$$ 

[Hint: Use induction.]

(c) If $\alpha$ is sufficiently small, show that $\sum_{n=0}^{\infty} x_n$ converges and $B\left(\sum_{n=0}^{\infty} x_n\right) = f$, and conclude that $B$ must also be surjective.
4. Let $S = S(\mathbb{R}^d)$ denote the Schwartz space and $\hat{f}$ denote the Fourier transform of $f \in S$.
(a) Prove that for $f, \phi \in S$,
$$
\lim_{\epsilon \to 0^+} \int f(x) \epsilon^{-d} \hat{\phi}(x/\epsilon) \, dx = f(0) \int \hat{\phi}(x) \, dx
$$
(b) Prove that for $f \in S$,
$$
f(x) = (2\pi)^{-d/2} \int \hat{f}(\xi) e^{ix \cdot \xi} \, d\xi.
$$

5. Let $H$ and $W$ be real Hilbert spaces and let $V \subset H$ be a linear subspace. Let $A : H \to H$ and $B : V \to W$ be bounded linear operators, where we give $V$ the norm $\|v\|_V = \|v\|_H + \|Bv\|_W$. For any $f \in V$ and $0 \leq \delta < 1$, consider the problem: Find $(u, p) \in V \times W$ such that
$$
\langle Au, v \rangle_H - \langle B^* p, v \rangle_H + \langle Bu, w \rangle_W + \langle p, w \rangle_W + \delta \langle Bu, Bv \rangle_W + \delta \langle p, Bv \rangle_W
= \langle f, v \rangle_H \quad \forall (v, w) \in V \times W.
$$
Assume that $A$ is coercive on $V$ (i.e., there is $\alpha > 0$ such that $\alpha \|v\|^2_V \leq \langle Av, v \rangle_V$).
(a) Assuming there is a solution, find a bound on the norm of the solution $(u, p)$.
(b) Show that there is a unique solution for any $\delta \in (0, 1)$.
(c) Show that there is a unique solution for $\delta = 0$. [Hint: Replace $w$ by $w - \delta Bv$.]

6. Let $X$ and $Y$ be normed vector spaces, and let $[a, b]$ and $(a, b)$ denote closed and open line segments between two given points $a, b \in X$.
(a) Let $f : X \to Y$ be a function which is continuous on the segment $[a, b]$ and differentiable on the segment $(a, b)$, and let $A \in B(X,Y)$ be given. Show that
$$
\|f(b) - f(a) - A(b-a)\|_Y \leq M\|b-a\|_X, \quad \text{where} \quad M = \sup_{x \in (a, b)} \|Df(x) - A\|_{B(X,Y)}.
$$
(b) Let $g : X \to Y$ be a function which is continuous in $X$ and differentiable in $X \setminus \{a\}$. Show that, if $L := \lim_{x \to a} Dg(x)$ exists, then $g$ is differentiable at $a$ and $Dg(a) = L$.
(c) Consider $g : X \to \mathbb{R}$ where $g(x) = \|x\|_X$. Show that $g$ cannot be differentiable at $x = 0$. 