(Note: Remember that a random variable $\xi$ is called a coin toss if $\mathbb{P}[\xi = 1] = \mathbb{P}[\xi = -1] = \frac{1}{2}$.)

Problem 1.

1. (Chernoff-Hoeffding bounds for coin tosses). Let $\{\xi_n\}_{n \in \mathbb{N}}$ be an iid sequence of coin tosses. Show that $E[e^{t\xi_1}] \leq \exp(\frac{1}{2}t^2)$ for all $t > 0$ and use it to prove that, for all $a > 0$ and $n \in \mathbb{N}$, we have

$$
\mathbb{P}\left[\sum_{i=1}^{n} \xi_i \geq a\right] \leq e^{-a^2/(2n)}
$$

and

$$
\mathbb{P}\left[\left|\sum_{i=1}^{n} \xi_i\right| \geq a\right] \leq 2e^{-a^2/(2n)}
$$

2. (Random matrices). Given $n, m \in \mathbb{N}$, let $A$ be an $n \times m$ matrix with entries in the set $\{0, 1\}$, and let $X$ be a random vector whose components $X = (X_1, \ldots, X_m)$ are independent coin tosses. Consider the random variable $||Y||_\infty = \max_{i=1,\ldots,n} |Y_i|$, where the vector $Y = (Y_1, \ldots, Y_n)$ is given by $Y = AX$. Show that

$$
\mathbb{P}\left[||Y||_\infty \geq \sqrt{4m \log n}\right] \leq \frac{2}{n}.
$$

(Hint: Establish an inequality of the same type for each $i = 1, \ldots, n$: consider separately the cases where the number of 1s in the $i$-th row of $A$ is above or below $\sqrt{4m \log n}$.)

Problem 2 (Randomness out of thin air). Let $\{(X_n, Y_n)\}_{n \in \mathbb{N}}$ be a sequence of random vectors (possibly defined on different probability spaces) converging weakly to the random vector $(X, Y)$. Show, by means of an example, that it is possible that $Y_n \in \sigma(X_n)$ for all $n \in \mathbb{N}$, but $Y \not\in \sigma(X)$. (Hint: Take $X_n$ uniform on $[0, 1)$ and let $Y_n$ be the $n$-th digit in the binary expansion of $X_n$.)

Problem 3 (Conditional expectation is not a projection in $L^1$). Let $X$ and $Y$ be two square-integrable random variables (on the same probability space).

1. If $f(x) = x^2$, show that $E\left[f\left(X - E[X|\sigma(Y)]\right)\right] \leq E\left[f(X - h(Y))\right]$ for any Borel $h : \mathbb{R} \to \mathbb{R}$ with $h(Y)$ square integrable.

2. Show that (1) above no longer holds true if we take $f(x) = |x|$. (Hint: A 3-element probability space will suffice.)