Problem 1. Let $X$ and $Y$ be two square-integrable random variables, defined on the same probability space, such that

$$\langle X(\omega') - X(\omega) \rangle \langle Y(\omega') - Y(\omega) \rangle \geq 0 \text{ for all } \omega, \omega' \in \Omega.$$ 

Show that $\text{Cov}(X, Y) \geq 0$.

Problem 2. A probability measure $\mu$ on (the Borel subsets of) $\mathbb{R}$ is said to be infinitely divisible, if, for each $n \in \mathbb{N}$, there exists a probability $\mu_n$ on $\mathbb{R}$ such that $\mu = \mu_n \ast \cdots \ast \mu_n$ ($n$-fold convolution).

(1) Show that any $\mu \in \mathbb{R}$ and $\sigma > 0$, the normal distribution, $N(\mu, \sigma^2)$ is infinitely divisible.

(2) Find another example of an infinitely divisible measure on $\mathbb{R}$.

(3) Find an example of a probability measure on $\mathbb{R}$ which is not infinitely divisible. (Hint: The sum of two discrete, independent and nonconstant random variables takes at least 4 different values with positive probabilities.)

(Note: The set of all probability measures on $\mathbb{R}$ admits the structure of (commutative) monoid with respect to the operation of convolution. Infinitely divisible measures are exactly those that admit an “$n$-th root” for each $n \in \mathbb{N}$.)

Problem 3. Given two independent simple symmetric random walks $\{\tilde{X}_n\}_{n \in \mathbb{N}_0}$ and $\{\tilde{Y}_n\}_{n \in \mathbb{N}_0}$, let $\{X_n\}_{n \in \mathbb{N}_0}$ denote $\{\tilde{X}_n\}_{n \in \mathbb{N}_0}$ stopped when it first hits the level 1, and let $\{Y_n\}_{n \in \mathbb{N}_0}$ be given by

$$Y_0 = 0, \quad Y_n = \sum_{k=1}^{n} 2^{-k}(\tilde{Y}_k - \tilde{Y}_{k-1}).$$

Identify the distribution of $\lim \inf_n (X_n + Y_n)$ and show that the sequence $\{X_n + Y_n\}_{n \in \mathbb{N}_0}$ is not uniformly integrable.