Directions: You have 90 minutes. Solve two of the three problems. Clearly mark which ones you want graded.

B1. Let $l$ and $p$ be primes. Show that the number of irreducible monic polynomials over $\mathbb{F}_p$, of degree $l$, is equal to $(p^l - p)/l$.

B2. Suppose $k$ is an algebraically closed field, $V$ is a finite-dimensional vector space over $k$, and $M : V \to V$ is a linear transformation. Show that there exists a unique pair of linear transformations $D, N : V \to V$ with the following properties. (For existence you can use well-known results, but for uniqueness you should argue directly.)

   1. $M = N + D$.
   2. $N$ is nilpotent, i.e. $N^s = 0$ for some integer $s > 0$.
   3. $D$ is diagonalizable, i.e. $V$ has a basis of $D$-eigenvectors.
   4. Every linear transformation $G$ commuting with $M$ also commutes with $N$ and $D$.

B3. Let $E$ be the splitting field of $x^7 - 3$ over $\mathbb{Q}$.

   (a) Determine the Galois group $\text{Gal}(E/\mathbb{Q})$ as a group of permutations of the roots of $x^7 - 3$.
   (b) Find a primitive generator of $E/\mathbb{Q}$.
   (c) Prove that $E$ is not a subfield of any cyclotomic extension of $\mathbb{Q}$.
   (d) Describe all the subfields of $E/\mathbb{Q}$ that are Galois over $\mathbb{Q}$. 