Work 3 of the following 4 problems.

1. Let $Y$ be a finite dimensional subspace of a normed linear space $X$. Prove that $Y$ is closed, and that there exists a continuous projection $P$ from $X$ onto $Y$. If $Y$ is one-dimensional, describe how to construct such a projection.

2. Let $X$ be a real Banach space with dual space $X'$ and duality pairing $\langle \cdot, \cdot \rangle$. Let $A, B : X \to X'$ be linear maps.
   (a) Assuming $\langle Ax, x \rangle \geq 0$ for all $x \in X$, show that $A$ is bounded.
   (b) Assuming $\langle Bx, y \rangle = \langle By, x \rangle$ for all $x, y \in X$, show that $B$ is bounded.

3. Let $\Omega = [a, b]$ and $1 < p, q < \infty$ be given, with $\frac{1}{p} + \frac{1}{q} = 1$. Let $v \in L^q(\Omega)$. For every $u \in L^p(\Omega)$ define a function $Au$ by setting

   $$(Au)(t) = \int_a^t v(s)u(s) \, ds, \quad \forall t \in \Omega.$$ 

   (a) Show that $A$ maps $L^p(\Omega)$ into $L^p(\Omega)$ and is continuous.
   (b) Show that $A : L^p(\Omega) \to L^p(\Omega)$ is compact.

4. Let $X$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $A : X \to X$ be a continuous linear map that satisfies $\langle Ax, x \rangle \geq 0$ for all $x \in X$. Show that
   (a) $\text{null}(A) = [\text{range}(A)]^\perp$.
   (b) $I + tA$ is bijective for every $t > 0$.
   (c) $\lim_{t \to \infty}(I + tA)^{-1}x = Px$ for all $x \in X$, where $P$ is the orthogonal projection in $X$ onto the null space $\text{null}(A)$ of $A$. 