Work all 3 of the following 3 problems.

1. Given $\alpha > 1$ and $\beta > 0$, consider the problem of finding a continuous function $u$ on $\Omega = [0, 1]$ that satisfies the equation

$$u(t) = \alpha + \beta \int_{0}^{t} s \ln |u(s)| \, ds, \quad \forall t \in \Omega.$$ 

Show that, if $\beta$ is sufficiently small, then this equation possesses a unique solution $u \in U$ in some open neighborhood $U \subset C(\Omega)$ of the constant function $t \mapsto \alpha$.

2. Let $X$ and $Y$ be normed linear spaces, and let $U \subset X$ be open. If $F : U \to Y$ is Gâteaux differentiable, and if the derivative $DF : U \to \mathcal{L}(X, Y)$ is continuous at $x \in U$, show that $F$ is Fréchet differentiable at $x$.

3. Let $\Omega$ be a domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. Let $A$ be a $n \times n$ matrix with components in $\text{L}^\infty(\Omega)$. Let $c \in \text{L}^\infty(\Omega)$ and $f \in \text{L}^2(\Omega)$. Consider the boundary value problem

$$-\nabla \cdot A \nabla u + cu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \quad (\star)$$ 

(All functions here are assumed to be real-valued.)

(a) Give the associated variational problem.

Assume now that $A$ is symmetric and uniformly positive definite, and that $c$ is uniformly positive. Define an energy functional $J : H^1_0(\Omega) \to \mathbb{R}$ by setting

$$J(u) = \frac{1}{2} \int_{\Omega} \left( |A^{1/2} \nabla u|^2 + c|u|^2 - 2fu \right), \quad \forall u \in H^1_0(\Omega).$$ 

(b) Compute the derivative $DJ(u)$.

(c) Prove that for $u \in H^1_0(\Omega)$ the following are equivalent: (i) $u$ is a weak solution of the boundary value problem $(\star)$, (ii) $DJ(u) = 0$, (iii) $u$ minimizes $J$. 