Preliminary Examination in Probability
Part II
August 24th, 2015

Problem 2.1. Consider two pairs of adapted continuous process \((H^i, X^i)\) defined on two filtered probability spaces \((\Omega_i, \mathcal{F}_i, (\mathcal{F}^i)_{0 \leq t < \infty}, \mathbb{P}_i)\) for \(i = 1, 2\). Assumed that the two pairs have the same law (as two-dimensional processes), and that \(X^1, X^2\) are semi-martingales. Show that the two stochastic integrals
\[
I^i = \int H^i dX^i
\]
have the same law for \(i = 1, 2\).

Note: we do not really need that \(H^i\) are continuous.

Problem 2.2. Consider a binary (i.e. which takes only two values) random variable \(X\) such that \(\mathbb{E}[X] = 0\). For a given Brownian motion \(B\), construct a stopping time (with respect to its natural filtration) with the property \(\mathbb{E}[T] < \infty\) and such that \(B_T\) and \(X\) have the same distribution. Is the condition \(\mathbb{E}[X] = 0\) necessary for the existence of such stopping time \(T\)?

Note: this is the simple instance of what is known as “Skorohod Imbedding”.

Problem 2.3. Show that, for a continuous semimartingale \(M\) and a continuous adapted process \(A\) of bounded variation, with \(A_0 = 0\), we have the following equivalence

(1) \(M\) is actually a local martingale and \(\langle M \rangle = A\),
(2) for each function \(f \in C^2_b\) (i.e. it is \(C^2\) and \(f\) together with its derivatives are bounded), we have that
\[
f(M_t) - f(M_0) - \int_0^t f''(M_s) dA_s
\]
is a martingale.