Preliminary Examination in Topology: August 2015
Algebraic Topology portion

Instructions: Do all three questions.

Time Limit: 90 minutes.

1. Consider the two 2-complexes $K$ and $L$ drawn below. $K$ is a punctured double torus with boundary curve $J$ with a disk added along a meridian curve. $L$ is a Klein bottle with a meridian curve $A$ drawn on it. Let $X$ be the complex obtained from $K \cup L$ by identifying $J$ with $A$ via a piecewise linear homeomorphism.
   
   a) What are the fundamental groups of $K$ and $L$? Briefly justify your answers.
   
   b) Using the decomposition of $X$ as $K \cup L$, compute the fundamental group of the complex $X$ using Van Kampen's Theorem.
   
   c) What are the homology groups of $K$ and $L$? Briefly justify your answers.
   
   d) Using the decomposition of $X$ as $K \cup L$, compute all the homology groups of the complex $X$ using the Mayer-Vietoris Theorem.

2. Describe all spaces that can be 4-fold covering spaces of the connected sum of five projective planes. Describe covering maps in each case with a picture. Explain why you know your list of spaces is complete.

3.

   a) Using insights from algebraic topology, prove that the free group on two elements has a normal subgroup of index three.

   b) Using insights from algebraic topology, prove that the free group on two elements has a subgroup of index three that is not normal.

   c) Are the two subgroups above necessarily free groups? If so, what are their ranks? Why?