Please try to solve 4 of the following 5 problems.

(1) For any \( r \geq 0 \) and any \( x \in \mathbb{R}^2 \) define \( B_r(x) = \{ y \in \mathbb{R}^2 : |y - x| \leq r \} \). Let \( 0 < c < 1 \). Let \( E \) be a measurable subset of the unit square \( Q \subset \mathbb{R}^2 \) with the property that for every \( x \in Q \) and every \( r > 0 \) there exists \( y \in B_r(x) \) such that \( B_{c|x-y|}(y) \subset E \). Prove that \( Q \setminus E \) has measure zero.

(2) Show that if \( p > 1 \) and \( f \in L^p([0, \infty), m) \) then the ‘mean functional’ of \( f \),
\[
F(y) := \frac{1}{y} \int_0^y f(t) \, dt = \int_0^1 f(xy) \, dx
\]
is also in \( L^p([0, \infty), m) \) and moreover
\[
\|F\|_p \leq \frac{p}{p - 1} \|f\|_p.
\]
Hint: consider \( f(xy) \) as a function of two variables on \([0, 1] \times [0, \infty)\) and use the generalized Minkowski inequality (which states that if \( g : X \times Y \to \mathbb{R} \) is any measurable function on the direct product of two sigma-finite measure spaces \((X, \mu), (Y, \nu)\) then
\[
\left\| \left\| g \right\|_{L^1(X, \mu)} \right\|_{L^p(Y, \nu)} \leq \left\| \left\| g \right\|_{L^p(Y, \nu)} \right\|_{L^1(X, \mu)}.
\]

(3) Let \((X, d)\) be a compact metric space. Let \( \{\mu_n\} \) be a sequence of positive Borel measures on \( X \) that converge in the weak* topology to a finite positive Borel measure \( \mu \). Show that for every compact \( K \subset X \),
\[
\mu(K) \geq \limsup_{n \to \infty} \mu_n(K).
\]

(4) Let \( 1 < p < \infty \). Assume \( f \in L^p(\mathbb{R}) \) satisfies
\[
\sup_{0 < |h| < 1} \left| \int \frac{f(x + h) - f(x)}{h} \, dx \right|^p dx < \infty.
\]
Show that \( f \) has a weak derivative \( g \in L^p \), which by definition satisfies \( \int \psi g = -\int \psi' f \) for every \( C^\infty \) function \( \psi \) on \( \mathbb{R} \) with compact support.

(5) Assuming \( f : [0, 1] \to \mathbb{R} \) is absolutely continuous, prove that \( f \) is Lipschitz if and only if \( f' \) belongs to \( L^\infty([0, 1]) \).