Please try to solve 4 of the following 5 problems.

(1) Suppose that $f$ is a holomorphic function on the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and that $f$ is injective on some annulus $\{ z \in \mathbb{C} : r < |z| < 1 \}$. Show that $f$ is injective on $\mathbb{D}$.

(2) Find all entire functions $f$ that satisfy $f(\sqrt{n}) = n^2$ for every positive integer $n$, and $|f(z)| \leq e^{3|z|}$ for every complex number $z$.

(3) Let $f_1, f_2, f_3, \ldots$ be analytic functions, defined on some domain $\Omega \subset \mathbb{C}$, and assume that $f_n \to f$ pointwise on $\Omega$. If none of the functions $f_n$ takes on any positive real values, show that $f$ is analytic on $\Omega$.

(4) Show that the equation $\sin(f(z)) = z$ has a solution $f$ that is analytic in the region $\Omega = \{ z \in \mathbb{C} : |z| < 1 \text{ or } \text{Im}(z) \neq 0 \}$.

(5) Consider the set $S$ of all analytic functions on $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ that are one-to-one and satisfy $f(0) = 0$ and $f'(0) = 1$. Show that if $f \in S$ then there exists an odd function $g \in S$ such that $g(z)^2 = f(z^2)$, for all $z \in \mathbb{D}$. 