1. Consider the ordinary differential equation initial value problem,

\[ x'(t) = f(x(t)), \quad t > 0, \]
\[ x(0) = x_0 \]

and the corresponding two stage Runge-Kutta approximation

\[ x_{n+1} = x_n + \alpha_1 h k_1 + \alpha_2 h k_2, \]
\[ k_1 = f(x_n), \quad k_2 = f(x_n + \beta h k_1), \]
\[ x_n = x(t_n), \quad t_n = nh \]

(a) For which \( \alpha_1, \alpha_2 \) and \( \beta \) will the method converge as \( h \to 0 \)?
(b) For which \( \alpha_1, \alpha_2 \) and \( \beta \) is the method of second order?
(c) Can a method on this form be A-stable?

Motivate your answers.

2. The following elliptic PDE is given,

\[ -\nabla \cdot a(x, y) \nabla u + b \cdot \nabla u + cu = f(x, y), \quad 0 < x < 1, \quad 0 < y < 1, \quad 0 < a \leq a(x, y) \leq A \]
\[ u = d_1(x, y), \quad x = 0 \text{ and } x = 1, \quad 0 < y < 1, \]
\[ u_y = d_2(x, y), \quad y = 0 \text{ and } y = 1, \quad 0 < x < 1, \]

(a) Rewrite the equation on weak form.
(b) Show that the relevant bilinear form is continuous and coercive and that the relevant linear form is continuous when \( d_1 = d_2 = 0 \) for appropriate values of the vector \( b \) and constant \( c > 0 \). Give the fundamental error estimate for a finite element approximation based on the weak form in terms of the best approximation in the space of basis function.
(c) Modify the boundary conditions to be appropriate for \( a(x, y) = 0 \) and describe a discontinuous Galerkin formulation for this case.
3. A hyperbolic system of nonlinear scalar conservation law has the form,

\[ u(x,t) + f_1(u(x,t))_x + g_1(u(x,t), v(x,t)) = 0 \]
\[ v(x,t) + f_2(v(x,t))_x + g_2(u(x,t), v(x,t)) = 0 \]

(a) Recommend suitable initial and boundary conditions for the hyperbolic system, \((t > 0, a < x < b)\).

(b) Devise an upwind finite difference method for the equation above when \(f_1(u) > 0, f_2(v) < 0\) and show that the method is consistent and for, \(g_1 = g_2 = 0\), on conservation form.

(c) Use von Neumann analysis when \(f_1(u) = a_1 u (a_1 > 0), f_2(u) = a_2 u (a_1 < 0)\), to determine necessary and sufficient conditions for the spatial and temporal step sizes, \(\Delta x, \Delta t\), to guarantee \(L^2\) stability.