1. Let $f \in H^1(\mathbb{R}^d)$ and $0 \leq s \leq 1$. Prove that there is a constant $C > 0$ such that
\[
\|f\|_{H^s(\mathbb{R}^d)} \leq C\|f\|_{H^1(\mathbb{R}^d)}^s \|f\|_{L^2(\mathbb{R}^d)}^{1-s}.
\]

2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial \Omega$. For $f(x), g(x)$ and $u_D(x)$ sufficiently smooth, consider the biharmonic boundary value problem (BVP)
\[
\begin{align*}
\Delta^2 u &= f \quad \text{in } \Omega, \\
\Delta u &= g \quad \text{on } \partial \Omega, \\
u &= u_D \quad \text{on } \partial \Omega.
\end{align*}
\]

(a) Reformulate the BVP as a variational problem for $u \in H^2(\Omega) \cap H^1_0(\Omega) + u_D$. Please indicate precisely the spaces in which the functions $f$, $g$, and $u_D$ lie.

(b) Apply the Lax-Milgram Theorem to show that there is a unique solution to the variational problem.

3. Let $\phi(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

(a) Consider the nonlinear initial value problem (IVP)
\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 3u^2 \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^2 \partial t} &= 0, \quad x \in \mathbb{R}, t > 0, \\
u(x, 0) &= \phi(x).
\end{align*}
\]

Use the Fourier transform in $x$ to show that the (IVP) can be rewritten in the form
\[
\begin{align*}
\partial_t u &= K \ast (u + u^3), \quad x \in \mathbb{R}, t > 0, \\
u(x, 0) &= \phi(x),
\end{align*}
\]
for some $K(x) \in L^1(\mathbb{R})$. [Hint: after rewriting the IVP in a convenient way (in particular, you can keep together the terms $u_x + 3u^2u_x$ by giving this group of terms a temporary name), apply the Fourier transform in $x$ to the IVP, simplify the expression that you obtain and then apply the inverse Fourier transform to formally obtain (1)–(2).]

(b) Set up and apply the contraction mapping principle to show that the initial value problem (1)–(2) has a continuous and bounded solution $u = u(x, t)$, at least up to some time $T < \infty$. 