PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part I
January 13, 2017, 1:00-2:30

Work all 3 of the following 3 problems.

1. Let $H$ be a Hilbert space and $P_j : H \to M_j$ be an orthogonal projection onto $M_j$, $j = 1, 2$. Let $N_j = N(P_j)$ be the nullspace of $P_j$.
   (a) Show that $\|P_j\| \leq 1$ and $P_j \geq 0$.
   (b) Show that the following are equivalent.
      i. $P_2P_1 = P_1P_2 = P_1$
      ii. $\|P_1x\| \leq \|P_2x\|$ for all $x \in H$
      iii. $P_1 \leq P_2$
      iv. $N_1 \supset N_2$
      v. $M_1 \supset M_2$

2. Let $X$ and $Y$ be Banach spaces. Let $A : X \to X^*$, $B : Y \to X^*$, and $C : Y \to Y^*$ be bounded linear operators. Suppose that $A$ maps onto $X^*$ and $C$ maps onto $Y^*$, and that there are constants $\alpha > 0$ and $\gamma > 0$ such that
   $Ax(x) \geq \alpha \|x\|^2_X$ and $Cy(y) \geq \gamma \|y\|^2_Y, \forall x \in X, y \in Y$.

Given $f \in X^*$ and $g \in Y^*$, consider the problem

$$Ax - By = f,$$
$$B^*x + Cy = g.$$ 

(a) The notation $B^*x$ is not quite correct. Explain its obvious meaning.
(b) Show that $A$ has an inverse and that $\|A^{-1}\| \leq 1/\alpha$.
(c) Prove that if there exists a solution $(x, y) \in X \times Y$ to the problem, then it is unique. [Hint: Show that $Ax(x) + Cy(y) = f(x) + g(y)$.] 
(d) If $\|B\| < \sqrt{\alpha \gamma}$, show that there is a solution to the problem.

3. Let $I = [0, 1]$ and $A : L^2(I) \to L^2(I)$ be defined by

$$Af(x) = \int_0^1 f(y) \sin \left(\frac{x + y}{2}\right) dy.$$ 

(a) Show that $A$ is compact and self-adjoint.
(b) Show that $\|A\| < 1$.
(c) Show that the smallest eigenvalue of $A$ is strictly negative. [Hint: Rayleigh Quotient.]