Planar fronts in bistable coupled map lattices

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The phase space is $[0, 1]^\mathbb{Z}^d$ and the map (CML) is

$$u_{t+1}^n = \sum_{s \in \mathbb{Z}^d} \ell_s f(u_{t-s}^n)$$

For each site $n$ in the lattice $\mathbb{Z}^d$ and for each discrete time $t \in \mathbb{Z}$ the variable $u^n_t \in [0, 1]$. The local map is a function $f : [0, 1] \to [0, 1]$. The convex diffusive coupling is given by the coefficients $\ell_s$ satisfying

$$\ell_s \geq 0 \quad \text{and} \quad \sum_{s \in \mathbb{Z}^d} \ell_s = 1.$$ 

**Example:** for $d = 2$, $\ell_{0,0} = 1 - \varepsilon$, $\ell_{1,0} = \ell_{-1,0} = \ell_{0,1} = \ell_{0,-1} = \frac{\varepsilon}{4}$ and $\ell_{m,n} = 0$ if $|m| + |n| > 1$.

$$u^{t+1}_{m,n} = (1 - \varepsilon) f(u^t_{m,n}) + \frac{\varepsilon}{4} (f(u^t_{m+1,n}) + f(u^t_{m+1,n+1}) + f(u^t_{m-1,n}) + f(u^t_{m,n-1}))$$

that is

$$u^{t+1} = f(u^t) + \frac{\varepsilon}{4} \Delta f(u^t)$$

where $\varepsilon \in [0, 1]$ is the coupling parameter.
**bistable map** on $[0, 1]$: a continuous increasing map $f : [0, 1] \rightarrow [0, 1]$
such that there exists $c \in (0, 1)$ so that

$$f(x) < x \text{ for all } x \in (0, c) \quad \text{and} \quad x < f(x) \text{ for all } x \in (c, 1).$$

Two stable fixed points: $f(0) = 0$ and $f(1) = 1$; one unstable fixed point: $f(c) = c$.

**Planar Fronts:** solutions $u^t_n$ of the form

$$u^t_n = \phi(n \cdot k - tv) \quad \forall t \in \mathbb{Z}$$

Where:

- $\phi : \mathbb{R} \rightarrow [0, 1]$ is the **front shape** satisfying $\lim_{x \to -\infty} \phi(x) = 0$ and $\lim_{x \to +\infty} \phi(x) = 1$
- $k$ is the direction of propagation ($k \in \mathbb{R}^d, \|k\| = 1$; i.e. $k \in S^{d-1}$)
- $v$ is the front velocity
For each direction $k \in S^{d-1}$ define the set

$$Z(k) = \{ \omega \in \mathbb{R} : \exists s \in \mathbb{Z}^d \quad \omega = s \cdot k \}.$$ 

and consider the invariant subset $\mathcal{X}_k \subset [0, 1]^\mathbb{Z}^d$, defined by

$$\mathcal{X}_k = \left\{ \{u_n\}_{n \in \mathbb{Z}^d} \in [0, 1]^\mathbb{Z}^d : m \cdot k = n \cdot k \Rightarrow u_m = u_n \right\}$$

For each $\{u_s\}_{s \in \mathbb{Z}^d} \in \mathcal{X}_k$ define the function $\psi \in [0, 1]^\mathbb{R}$, in $Z(k)$ by

$$\psi(s \cdot k) = u_s$$

and for $x \in \mathbb{R} \setminus Z(k)$ by $\psi(x) = 0$ (0 being a fixed point of $f$).

Then the dynamics of the (CML) implies that

$$\psi^{t+1}(x) = \sum_{s \in \mathbb{Z}^d} \ell_s f \circ \psi^t(x - s \cdot k).$$

Hence the (CML) dynamics induces a map $F : [0, 1]^\mathbb{R} \to [0, 1]^\mathbb{R}$. $\psi^{t+1} = F(\psi^t)$.

- If the direction $k$ is totally irrational ($\forall s \in \mathbb{Z}^d \quad s \cdot k = 0 \Rightarrow s = 0$), then the (CML) dynamics is a restriction of the dynamics given by the map $F$, because $\mathcal{X}_k = [0, 1]^\mathbb{Z}^d$ in this case.

- If the direction $k$ is not totally irrational, then the map $F$ is an extension of the (CML) dynamics restricted to the set $\mathcal{X}_k$. 
If we restrict the map \( F(\psi)(x) = \sum_{s \in \mathbb{Z}^d} \ell_s f \circ \psi(x - s \cdot k) \) to the set \( B \) of Borel measurable functions on \( \mathbb{R} \) with values in \([0, 1]\), then the map \( F \) can be written as

\[
F(\psi) = h_k \ast f \circ \psi,
\]

where the convolution is defined by the Lebesgue-Stieltjes integral

\[
h \ast \phi(x) = \int_{\mathbb{R}} \phi(x - y) \, dh(y)
\]

and the distribution function \( h_k : \mathbb{R} \rightarrow [0, 1] \) is

\[
h_k(x) = \sum_{s \in \mathbb{Z}^d} \ell_s H(x - s \cdot k)
\]

and \( H \) is the Heaviside function:

\[
H(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x \geq 0
\end{cases}
\]

**Reformulating the problem:**

Given a distribution function \( h : \mathbb{R} \rightarrow [0, 1] \) (increasing function with the following limits \( \lim_{x \rightarrow -\infty} h(x) = 0 \) and \( \lim_{x \rightarrow +\infty} h(x) = 1 \)) and a bistable map \( f : [0, 1] \rightarrow [0, 1] \) define the map \( F : B \rightarrow B \) by

\[
F(\psi) = h \ast f \circ \psi
\]

Find a front shape \( \phi \in B \) and a velocity \( v \), satisfying:

\[
T^v \phi = F(\phi) , \text{ with } \lim_{x \rightarrow -\infty} \phi(x) = 0 \text{ and } \lim_{x \rightarrow +\infty} \phi(x) = 1
\]

where \( T^v \) is the **translation** by \( v \in \mathbb{R} \) defined by \( T^v u(x) = u(x - v) \).
Results

Existence of fronts

**Theorem** For any distribution function $h$ and any bistable map $f$, there exists a velocity $v \in \mathbb{R}$ and an increasing function $\phi$ such that

$$
\lim_{x \to -\infty} \phi(x) = 0, \quad \lim_{x \to +\infty} \phi(x) = 1 \quad \text{and} \quad T^v \phi = F\phi,
$$

where $F\phi = h * f \circ \phi$.

Uniqueness of front velocity

**Theorem** For any distribution function $h$ and any bistable regular map $f$, there exists a unique velocity $v \in \mathbb{R}$ and an increasing function $\phi$ such that

$$
\lim_{x \to -\infty} \phi(x) = 0, \quad \lim_{x \to +\infty} \phi(x) = 1 \quad \text{and} \quad T^v \phi = F\phi,
$$

where $F\phi = h * f \circ \phi$.

Bistable regular maps

A bistable map is said to be regular if it is a weak contraction in a neighbourhood of each stable fixed point. That is

$$
\exists \delta > 0 \quad \left[ x, y \in (0, \delta) \quad \text{or} \quad x, y \in (1 - \delta, 1) \right] \quad \Rightarrow \quad |f(x) - f(y)| \leq |x - y|.
$$
Using the Taylor’s formula we find the following sufficient conditions for a bistable map $f$ to be regular:

a) $f$ analytic or b) $f \in C^1$ and $f'(0) < 1$ and $f'(1) < 1$ or $f \in C^2$ and $f''(0) \neq 0$ and $f''(1) \neq 0$ or d) $f \in C^3$ and $f'''(0) \neq 0$ and $f'''(1) \neq 0$ or ...

Nevertheless it is possible to construct $C^\infty$ bistable maps that are not regular.
Continuity of the front velocity

Assume that $f$ is \textbf{regular} and let $v(f, h)$ be the unique front velocity of $F$.

**Theorem** Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of regular bistable maps which converges pointwise to a bistable regular map $f$. Let $\{h_n\}_{n \in \mathbb{N}}$ be a sequence of distribution functions and $h$ be a distribution function such that $\lim_{n \to \infty} d(h_n, h) = 0$. Then $\lim_{n \to \infty} v(f_n, h_n) = v(f, h)$.

Given two right continuous distribution functions $h$ and $h'$, define the \textbf{Lévy distance}

$$d(h, h') = \inf \{\varepsilon > 0 : h(x - \varepsilon) - \varepsilon \leq h'(x) \leq h(x + \varepsilon) + \varepsilon, \quad \forall x \in \mathbb{R}\}.$$

This is the same as the \textbf{Hausdorff distance} restricted to graphs of distribution functions. \textbf{Hausdorff distance}:

$$d(h, h') = \max \left\{ \sup_{z_1 \in Gh} \inf_{z_2 \in Gh'} \|z_1 - z_2\|, \sup_{z_1 \in Gh'} \inf_{z_2 \in Gh} \|z_1 - z_2\| \right\},$$

where $Gh$ is the graph of $h$, i.e. $Gh = \{(x, y) : h(x^-) \leq y \leq h(x^+)\}$, and the $\mathbb{R}^2$ norm $\| \cdot \|$ is given by $\|(x, y)\| = \max \{|x|, |y|\}$. 
Recall that $c$ denotes the unstable fixed point of $f$.

An **interface** is a function $u \in B$ if there exists $c_- \in (0, c)$, $c_+ \in (c, 1)$ and $j_1 \leq j_2 \in \mathbb{R}$ so that

$$u(x) \leq c_- \quad \text{if} \quad x \leq j_1 \quad \text{and} \quad u(x) \geq c_+ \quad \text{if} \quad x \geq j_2.$$
Given the level \( a \in (0, 1) \), the **reference center** of a function \( u \) is

\[
J_a(u) = \inf \{ x \in \mathbb{R} : u(x) \geq a \}.
\]

By applying a translation a function \( u \) can be centered at \( 0 \):

\[
J_a(T^{-J_a(u)}u) = 0.
\]

**Velocity of interfaces**

**Theorem** Let \( h \) be a distribution function and let \( f \) be a regular bistable map. For every interface \( u \) and every \( a \in (0, 1) \), we have

\[
\lim_{t \to \infty} \frac{J_a(F^t u)}{t} = v(f, h).
\]

Where \( v(f, h) \) is the (unique) front velocity of \( F \).
Application to the planar fronts of (CML)

\[ u_{n+1}^t = \sum_{s \in \mathbb{Z}^d} \ell_s f(u_{n-s}^t), \quad \text{with } f \text{ bistable, } \ell_s \geq 0 \text{ and } \sum_{s \in \mathbb{Z}^d} \ell_s = 1. \]

If \( \phi \) is a function such that \( \lim_{x \to -\infty} \phi(x) = 0, \lim_{x \to +\infty} \phi(x) = 1 \) and \( T^v \phi = F \phi \), where \( F \phi = h_k * f \circ \phi \), then defining

\[ u_n^t = \phi(\sigma + n \cdot k - tv) \]

(for an arbitrary phase \( \sigma \in \mathbb{R} \)), we have \( u_{n+1}^t = \sum_{s \in \mathbb{Z}^d} \ell_s f(u_{n-s}^t) \), i.e. \( u_n^t \) is a planar front with direction of propagation \( k \) and front shape \( \phi \).

Existence of planar fronts

**Theorem** For any direction \( k \in S^{d-1} \) and any bistable function \( f \) there exists planar fronts in the direction \( k \) for (CML). If \( f \) is regular, then the velocity of these fronts is uniquely determined for each direction \( k \).

The velocity of fronts \( v \) depends then on \( f, \{\ell_s\}_{s \in \mathbb{Z}^d} \) and \( k \):

\[ v = v(f, \{\ell_s\}_{s \in \mathbb{Z}^d}, k) \]
Continuity of the velocity

**Theorem** Given $\epsilon > 0$ there exists $\delta > 0$ such that for any $f, f'$ regular bistable, $\{\ell_s\}_{s \in \mathbb{Z}^d}, \{\ell'_s\}_{s \in \mathbb{Z}^d}$ nonnegative and normalized and $k, k' \in S^{d-1}$ satisfying

$$\|k - k'\| < \delta, \quad \sum_{s \in \mathbb{Z}^d} |\ell'_s - \ell_s| < \delta \quad \text{and} \quad \sup |f - f'| < \delta,$$

we have

$$|v(f, \{\ell_s\}_{s \in \mathbb{Z}^d}, k) - v(f', \{\ell'_s\}_{s \in \mathbb{Z}^d}, k')| < \epsilon.$$

**Interfaces**

A configuration $\{u_s\}_{s \in \mathbb{Z}^d} \in [0, 1]^{\mathbb{Z}^d}$ is an interface in the direction $k$ if there exists $j_-, j_+, 0 < c_- < c < c_+ < 1$, such that

$$n \cdot k < j_- \Rightarrow u_n < c_- \quad \text{and} \quad n \cdot k > j_+ \Rightarrow u_n > c_+.$$

(note that $n \cdot k$ measures a position along the line orthogonal to $k$)

**Theorem** If $\{u_0^s\}_{s \in \mathbb{Z}^d, t \in \mathbb{N}}$ is an interface in the direction $k$, then the evolution $\{u'_s\}_{s \in \mathbb{Z}^d}$ by (CML) is an interface in the direction $k$ and

$$\lim_{t \to +\infty} \frac{1}{t} J_a^k(\{u'_s\}_{s \in \mathbb{Z}^d}) = v(f, \{\ell_s\}_{s \in \mathbb{Z}^d}, k).$$

Where $J_a^k(\{u_s\}_{s \in \mathbb{Z}^d}) = \inf \{ j \in \mathbb{R} : n \cdot k > j \Rightarrow u_n \geq a \}$ and $a \in (0, 1)$. 
Extended bistable maps

$$u^{t+1} = Fu^t := h * f \circ u^t$$

The phase space is the set $\mathcal{B}$ of Borel-measurable functions on $\mathbb{R}$ with values in $[0, 1]$.

**Basic properties:**

**Homogeneity**

$$T^v F = FT^v \quad \text{for all } v \in \mathbb{R}.$$  

**Continuity:**

$$\forall x \in \mathbb{R} \quad \lim_{n \to \infty} u_n(x) = u(x) \quad \Rightarrow \quad \forall x \in \mathbb{R} \quad \lim_{n \to \infty} Fu_n(x) = Fu(x).$$

**Monotony:**

$$u \leq v \quad \Rightarrow \quad Fu \leq Fv.$$
Sketch of the proof of existence of fronts

Let $\mathcal{I} \subset \mathcal{B}$ be the subset composed of increasing functions, $v \in \mathbb{R}$ and $c_+ \in (c, 1)$. The set $S_{v,c_+}$ of **sub-fronts** of velocity $v$:

$$S_{v,c_+} = \{ \psi \in \mathcal{I} : F\psi \leq T^v\psi \text{ and } J_{c_+}(\psi) = 0 \}.$$

When $S_{v,c_+}$ is not empty, consider the function

$$\eta_v(x) = \inf_{\psi \in S_{v,c_+}} \psi(x), \quad x \in \mathbb{R}.$$

It turns out that $\eta_v \in S_{v,c_+}$ and therefore $\eta_v$ is a **minimal sub-front** of velocity $v$.

We also prove the existence of a maximal sub-fronts velocity $\bar{v} = \max \{ v \in \mathbb{R} : S_{v,c_+} \neq \emptyset \}$.

Consider the reference centers of the iterates $F^n\eta_{\bar{v}}$ of the minimal sub-front $\eta_{\bar{v}}$ for the maximal sub-fronts velocity $\bar{v}$:

$$j_n := J_{c_+}(F^n\eta_{\bar{v}})$$

Then we prove that $\lim \inf_{n \to \infty} (j_{n+m} - j_n) = m\bar{v}$.

From this we use an arithmetical Lemma that ensures that there exists a strictly increasing sequence $\{n_k\}$ such that for all $m$

$$\lim_{k \to \infty} (j_{n_k+m} - j_{n_k}) = m\bar{v}.$$
Using this subsequence \( \{n_k\} \), we consider the sequence \( \{T^{-j_n} F^{n_k} \eta \}_k \) from which a convergent subsequence can be extracted by Helly’s Selection Theorem:

\[
\eta_\infty = \lim_{k \to \infty} T^{-j_n} F^{n_k} \eta.
\]

Consider now the sequence \( \{T^{-m\tilde{v}} F^m \eta_\infty\}_k \). It satisfies \( \eta \leq T^{-(m+1)\tilde{v}} F^{m+1} \eta_\infty \leq T^{-m\tilde{v}} F^m \eta_\infty \). Hence, the following limit exists

\[
\phi = \lim_{m \to \infty} T^{-m\tilde{v}} F^m \eta_\infty
\]

and satisfies \( T^{\tilde{v}} \phi = F \phi \) and \( \lim \phi(x) = 1 \). As for the limit \( \lim \phi(x) \), in general we cannot say more than \( \lim \phi(x) \in \{0, c\} \).

However, if \( f \) and \( h \) are such that

\[
\frac{df}{dc}(c) = +\infty \quad \text{and} \quad \inf \{x \in \mathbb{R} : h(x) > 0\} = -\infty,
\]

then one can prove that \( \lim \phi(x) = 0 \).

Finally, to conclude in the general case, we show that every pair of bistable map \( f \) and distribution function \( h \) can be approximated pairs satisfying the previous condition. The existence of fronts then follows from continuity properties (see the references for more details).