# Integrable billiards, Poncelet-Darboux grids and Kowalevski top

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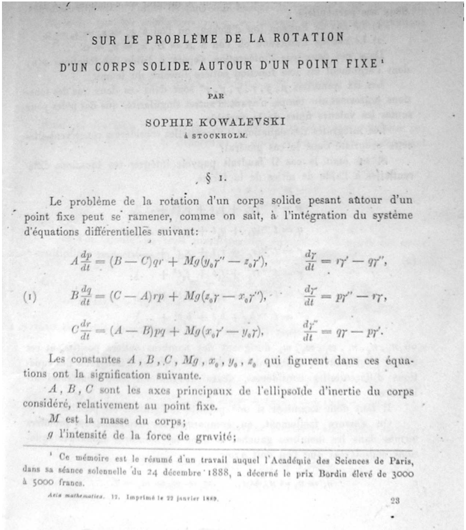
# Sophie's world

Софья Васильевна Ковалевская, 1850–1891

# Sophie's world: the story of the history of integrability



SUR LE PROBLÈME DE LA ROTATION

D'UN CORPS SOLIDE AUTOUR D'UN POINT FIXE[1]

PAR

SOPHIE KOWALEVSKI

À STOCKHOLM.

§ 1.

Le problème de la rotation d'un corps solide pesant autour d'un point fixe peut se ramener, comme on sait, à l'intégration du système d'équations différentielles suivant:

$$(1)\qquad \begin{aligned} A\frac{dp}{dt} &= (B-C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'',\\ B\frac{dq}{dt} &= (C-A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma,\\ C\frac{dr}{dt} &= (A-B)pq + Mg(x_0\gamma' - y_0\gamma), & \frac{d\gamma''}{dt} &= q\gamma - p\gamma'. \end{aligned}$$

Les constantes $A$, $B$, $C$, $Mg$, $x_0$, $y_0$, $z_0$ qui figurent dans ces équations ont la signification suivante.

$A$, $B$, $C$ sont les axes principaux de l'ellipsoïde d'inertie du corps considéré, relativement au point fixe.

$M$ est la masse du corps;

$g$ l'intensité de la force de gravité;

[1] Ce mémoire est le résumé d'un travail auquel l'Académie des Sciences de Paris, dans sa séance solennelle du 24 décembre 1888, a décerné le prix Bordin élevé de 3000 à 5000 francs.



Acta Mathematica, 1889

# Addition Theorems

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\mathrm{sn}\,(x+y) = \frac{\mathrm{sn}\, x\ \mathrm{cn}\, y\ \mathrm{dn}\, y + \mathrm{sn}\, y\ \mathrm{cn}\, x\ \mathrm{dn}\, x}{1 - k^2\ \mathrm{sn}^2 x\ \mathrm{sn}^2 y}$$
$$\mathrm{cn}\,(x+y) = \frac{\mathrm{cn}\, x\ \mathrm{cn}\, y - \mathrm{sn}\, x\ \mathrm{sn}\, y\ \mathrm{dn}\, x\ \mathrm{dn}\, y}{1 - k^2\ \mathrm{sn}^2 x\ \mathrm{sn}^2 y}$$

$$\wp(x+y) = -\wp(x) - \wp(y) + \frac{1}{4}\left(\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)}\right)^2$$

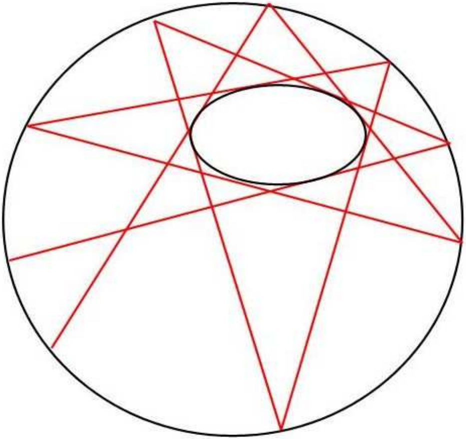
# The Quantum Yang-Baxter Equation

$$R^{12}(t_1-t_2,h)R^{13}(t_1,h)R'^{23}(t_2,h) = R^{23}(t_2,h)(R^{13}(t_1,h)R^{12}(t_1-t_2,h)$$

$$R^{ij}(t,h) : V \otimes V \otimes V \to V \otimes V \otimes V$$

$t$ – spectral parameter
$h$ – Planck constant

# The Euler-Chasles correspondence



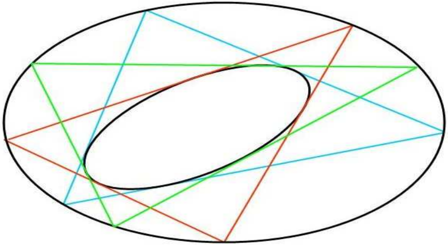
$$E : ax^2y^2 + b(x^2y + xy^2) + c(x^2 + y^2) + 2dxy + e(x + y) + f = 0$$

# The Euler-Chasles correspondence

## The Euler equation

$$\frac{dx}{\sqrt{p_4(x)}} \pm \frac{dy}{\sqrt{p_4(y)}} = 0$$

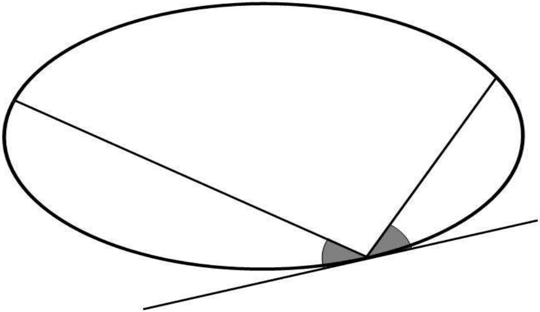
## Poncelet theorem for triangles



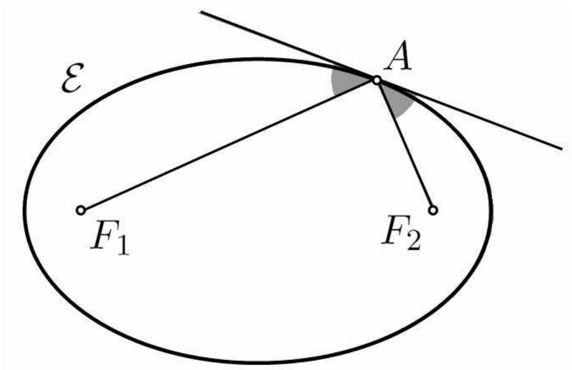
# Elliptical Billiard

# Billiard within ellipse

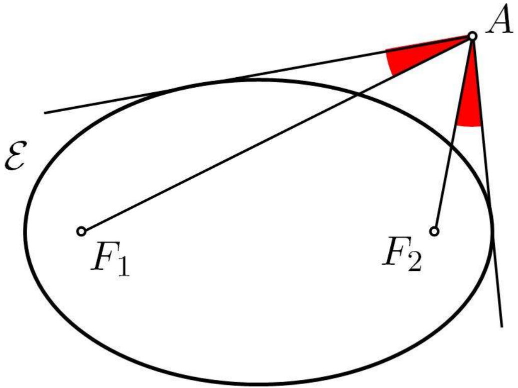
A trajectory of a billirad within an ellipse is a polygonal line with vertices on the ellipse, such that successive edges satisfy the billiard reflection law: the edges form equal angles with the to the ellipse at the common vertex.



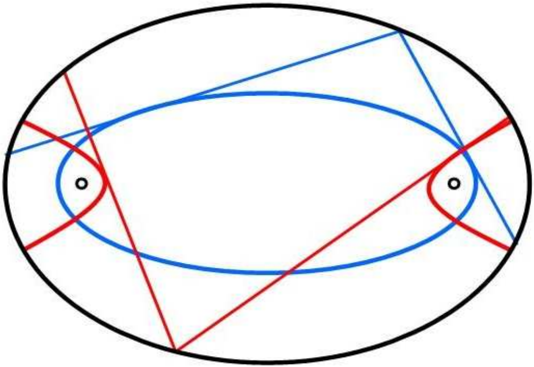
# Focal property of ellipses



# Focal property of ellipses

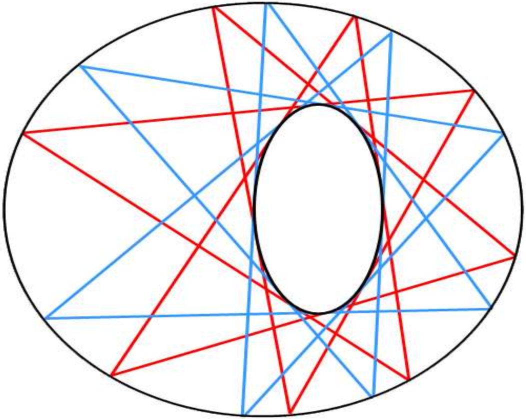


# Caustics of billiard trajectories



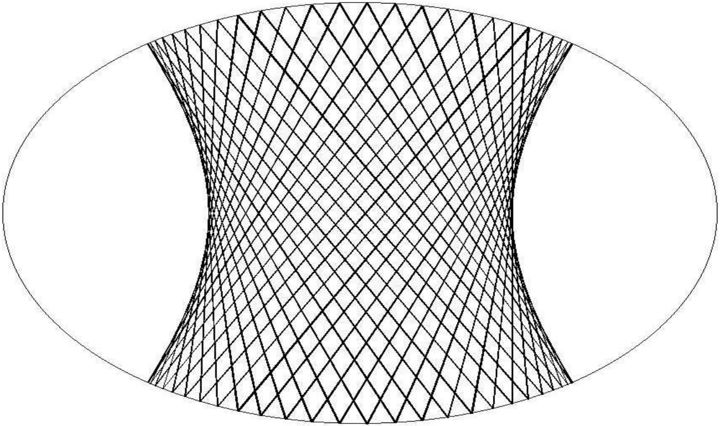
# Poncelet theorem (Jean Victor Poncelet, 1813.)

Let $\mathcal{C}$ and $\mathcal{D}$ be two given conics in the plane. Suppose there exists a closed polygonal line inscribed in $\mathcal{C}$ and circumscribed about $\mathcal{D}$. Then, there are infinitely many such polygonal lines and all of them have the same number of edges. Moreover, every point of the conic $\mathcal{C}$ is a vertex of one of these lines.

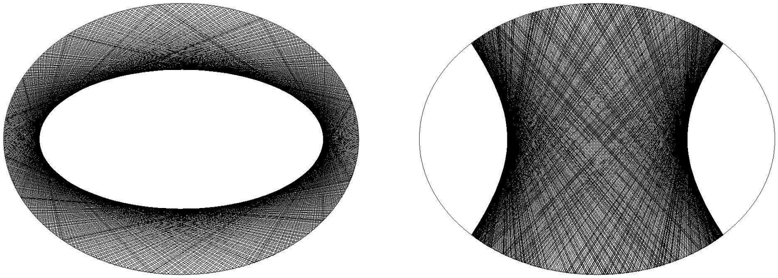


# Mechanical interpretation of the Poncelet theorem

Let us consider closed trajectory of billiard system within ellipse $\mathcal{E}$. Then every billiard trajectory within $\mathcal{E}$, which has the same caustic as the given closed one, is also closed. Moreover, all these trajectories are closed with the same number of reflections at $\mathcal{E}$.



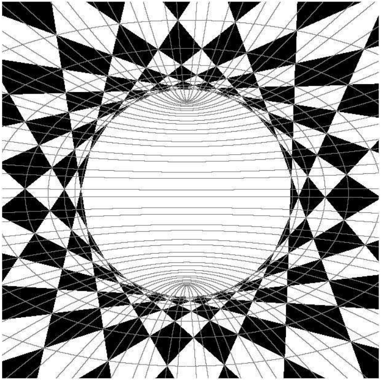
# Nonperiodic billiard trajectories



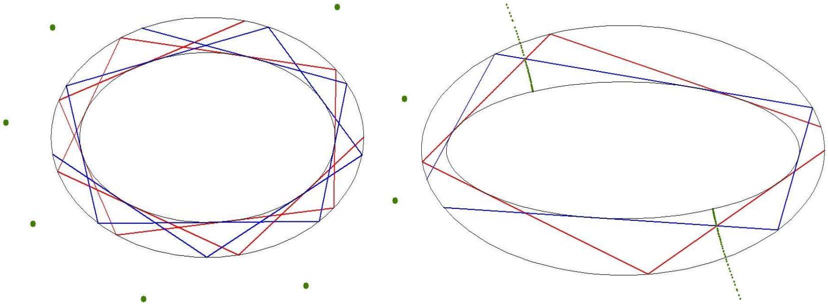
# Generalization of the Darboux theorem

**Theorem [V. D., M. Radnović (2008)]**

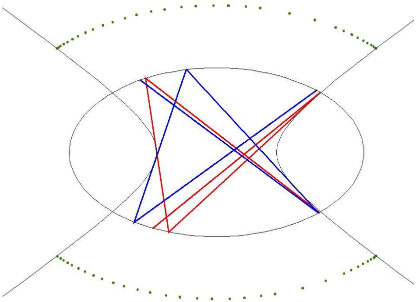
Let $\mathcal{E}$ be an ellipse in $\mathbf{E}^2$ and $(a_m)_{m\in\mathbf{Z}}$, $(b_m)_{m\in\mathbf{Z}}$ be two sequences of the segments of billiard trajectories $\mathcal{E}$, sharing the same caustic. Then all the points $a_m \cap b_m$ $(m \in \mathbf{Z})$ belong to one conic $\mathcal{K}$, confocal with $\mathcal{E}$.

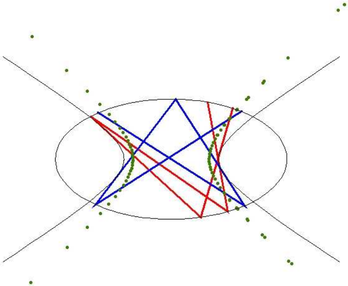


Moreover, under the additional assumption that the caustic is an ellipse, we have:
if both trajectories are winding in the same direction about the caustic, then $\mathcal{K}$ is also an ellipse; if the trajectories are winding in opposite directions, then $\mathcal{K}$ is a hyperbola.



For a hyperbola as a caustic, it holds:
if segments $a_m$, $b_m$ intersect the long axis of $\mathcal{E}$ in the same direction, then $\mathcal{K}$ is a hyperbola, otherwise it is an ellipse.





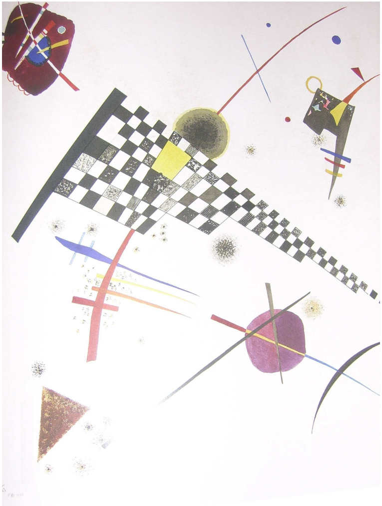
# Grids in arbitrary dimension

**Theorem [V. D., M. Radnović (2008)]**

Let $(a_m)_{m\in\mathbf{Z}}$, $(b_m)_{m\in\mathbf{Z}}$ be two sequences of the segments of billiard trajectories within the ellipsoid $\mathcal{E}$ in $\mathbf{E}^d$, sharing the same $d-1$ caustics. Suppose the pair $(a_0, b_0)$ is $s$-skew, and that by the sequence of reflections on quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$ the minimal billiard trajectory connecting $a_0$ to $b_0$ is realized.

Then, each pair $(a_m, b_m)$ is $s$-skew, and the minimal billiard trajectory connecting these two lines is determined by the sequence of reflections on the same quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$.

# Kandinsky, *Grid* 1923.



# Pencil of conics

## Two conics and tangential pencil

$$C_1 \; : \; a_0 w_1^2 + a_2 w_2^2 + a_4 w_3^2 + 2a_3 w_2 w_3 + 2a_5 w_1 w_3 + 2a_1 w_1 w_2 = 0$$
$$C_2 \; : \; w_2^2 - 4 w_1 w_3 = 0$$

## Coordinate pencil

$$F(s, z_1, z_2, z_3) := \det M(s, z_1, z_2, z_3) = 0$$
$$M(s, z_1, z_2, z_3) = \begin{bmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & a_0 & a_1 & a_5 - 2s \\ z_2 & a_1 & a_2 + s & a_3 \\ z_3 & a_5 - 2s & a_3 & a_4 \end{bmatrix}$$
$$F := H + Ks + Ls^2 = 0$$

# Darboux coordinates

$$t_{C_2}(\ell_0) : z_1\ell_0^2 - 2z_2\ell_0 + z_3 = 0$$
$$\hat{z}_1\ell^2 - 2\hat{z}_2\ell + \hat{z}_3 = 0$$
$$\hat{z}_1 = 1, \quad \hat{z}_2 = \frac{x_1 + x_2}{2}, \quad \hat{z}_3 = x_1x_2$$

$$F(s, x_1, x_2) = L(x_1, x_2)s^2 + K(x_1, x_2)s + H(x_1, x_2)$$
$$\begin{aligned}H(x_1, x_2) = {} & (a_1^2 - a_0a_2)x_1^2x_2^2 + (a_0a_3 - a_5a_1)x_1x_2(x_1 + x_2) \\ & + (a_5^2 - a_0a_4)(x_1^2 + x_2^2) + (2(a_5a_2 - a_1a_3) + \frac{1}{2}(a_5^2 - a_0a_4)x_1x_2 \\ & + (a_1a_4 - a_3a_5))(x_1 + x_2) + a_3^2 - a_2a_4\end{aligned}$$
$$\begin{aligned}K(x_1, x_2) = {} & -a_0x_1^2x_2^2 + 2a_1x_1x_2(x_1 + x_2) - a_5(x_1^2 + x_2^2) - 4a_2x_1x_2 \\ & + 2a_3(x_1 + x_2) - a_4\end{aligned}$$
$$L(x_1, x_2) = (x_1 - x_2)^2$$

## Theorem [V. D. (2009)]

(i) There exists a polynomial $P = P(x)$ such that the discriminant of the polynomial $F$ in $s$ as a polynomial in variables $x_1$ and $x_2$ separates the variables:

$$\mathcal{D}_s(F)(x_1, x_2) = P(x_1)P(x_2). \tag{1}$$

(ii) There exists a polynomial $J = J(s)$ such that the discriminant of the polynomial $F$ in $x_2$ as a polynomial in variables $x_1$ and $s$ separates the variables:

$$\mathcal{D}_{x_2}(F)(s, x_1) = J(s)P(x_1). \tag{2}$$

Due to the symmetry between $x_1$ and $x_2$ the last statement remains valid after exchanging the places of $x_1$ and $x_2$.

### Lemma

Given a polynomial $S = S(x, y, z)$ of the second degree in each of its variables in the form:

$$S(x, y, z) = A(y, z)x^2 + 2B(y, z)x + C(y, z).$$

If there are polynomials $P_1$ and $P_2$ of the fourth degree such that

$$B(y, z)^2 - A(y, z)C(y, z) = P_1(y)P_2(z), \tag{3}$$

then there exists a polynomial $f$ such that

$$\mathcal{D}_y S(x, z) = f(x)P_2(z), \quad D_z S(x, y) = f(x)P_1(y).$$

# Gauge equivalence

## Gauge transformations

$$x \mapsto \frac{a_1 x + b_1}{c_1 x + d_1}$$

$$y \mapsto \frac{a_2 y + b_2}{c_2 y + d_2}$$

$$z \mapsto \frac{a_3 z + b_3}{c_3 z + d_3}$$

# Discriminantly separable polynomials – definition

For a polynomial $F(x_1, \ldots, x_n)$ we say that it is discriminantly separable if there exist polynomials $f_i(x_i)$ such that for every $i = 1, \ldots, n$

$$\mathcal{D}_{x_i} F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j(x_j).$$

It is symmetrically discriminantly separable if

$$f_2 = f_3 = \cdots = f_n,$$

while it is strongly discriminatly separable if

$$f_1 = f_2 = f_3 = \cdots = f_n.$$

It is weakly discriminantly separable if there exist polynomials $f_i^j(x_i)$ such that for every $i = 1, \ldots, n$

$$\mathcal{D}_{x_i} F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j^i(x_j).$$

### Theorem [V. D. (2009)]

Given a polynomial $F(s, x_1, x_2)$ of the second degree in each of the variables $s, x_1, x_2$ of the form

$$F = s^2 A(x_1, x_2) + 2B(x_1, x_2)s + C(x_1, x_2).$$

Denote by $T_{B^2-AC}$ a $5 \times 5$ matrix such that

$$(B^2 - AC)(x_1, x_2) = \sum_{j=1}^{5} \sum_{i=1}^{5} T^{ij}_{B^2-AC} x_1^{i-1} x_2^{j-1}.$$

Then, polynomial $F$ is discriminantly separable if and only if

$$\mathrm{rank}\, T_{B^2-AC} = 1.$$

# Geometric interpretation of the Kowalevski fundamental equation

$$Q(w, x_1, x_2) := (x_1 - x_2)^2 w^2 - 2R(x_1, x_2)w - R_1(x_1, x_2) = 0$$

$$\begin{aligned} R(x_1, x_2) = & -x_1^2 x_2^2 + 6\ell_1 x_1 x_2 + 2\ell c(x_1 + x_2) + c^2 - k^2 \\ R_1(x_1, x_2) = & -6\ell_1 x_1^2 x_2^2 - (c^2 - k^2)(x_1 + x_2)^2 - 4c\ell x_1 x_2 (x_1 + x_2) \\ & + 6\ell_1 (c^2 - k^2) - 4c^2 \ell^2 \end{aligned}$$

$$\begin{aligned} a_0 = & -2 \qquad a_1 = 0 \qquad a_5 = 0 \\ a_2 = & \; 3\ell_1 \qquad a_3 = -2c\ell \qquad a_4 = 2(c^2 - k^2) \end{aligned}$$

# Geometric interpretation of the Kowalevski fundamental equation

### Theorem

[V. D. (2009)] The Kowalevski fundamental equation represents a point pencil of conics given by their tangential equations

$$\hat{C}_1 : -2w_1^2 + 3l_1 w_2^2 + 2(c^2 - k^2)w_3^2 - 4clw_2w_3 = 0;$$
$$C_2 : w_2^2 - 4w_1w_3 = 0.$$

The Kowalevski variables $w, x_1, x_2$ in this geometric settings are the pencil parameter, and the Darboux coordinates with respect to the conic $C_2$ respectively.

# Multi-valued Buchstaber-Novikov groups

## $n$-valued group on $X$

$$m \;:\; X \times X \to (X)^n, \qquad m(x,y) = x * y = [z_1, \ldots, z_n]$$

$(X)^n$ — symmetric $n$-th power of $X$

## Associativity

Equality of two $n^2$-sets:

$$[x * (y * z)_1, \ldots, x * (y * z)_n] \quad \text{и} \quad [(x * y)_1 * z, \ldots, (x * y)_n * z]$$

for every triplet $(x, y, z) \in X^3$.

## Unity $e$

$e * x = x * e = [x, \ldots, x]$ for each $x \in X$.

## Inverse $\mathrm{inv} : X \to X$

$e \in \mathrm{inv}(x) * x$, $e \in x * \mathrm{inv}(x)$ for each $x \in X$.

# Multi-valued Buchstaber-Novikov groups

## Action of $n$-valued group $X$ on the set $Y$

$$\phi \; : \; X \times Y \to (Y)^n$$
$$\phi(x, y) = x \circ y$$

Two $n^2$-multi-subsets in $Y$:

$$x_1 \circ (x_2 \circ y) \quad \text{и} \quad (x_1 * x_2) \circ y$$

are equal for every triplet $x_1, x_2 \in X$, $y \in Y$.
Additionally , we assume:

$$e \circ y = [y, \dots, y]$$

for each $y \in Y$.

# Two-valued group on $\mathbf{CP}^1$

## The equation of a pencil

$F(s, x_1, x_2) = 0$

## Isomorphic elliptic curves

$$\Gamma_1 \ : \ y^2 = P(x) \qquad \deg P = 4$$
$$\Gamma_2 \ : \ t^2 = J(s) \qquad \deg J = 3$$

## Canonical equation of the curve $\Gamma_2$

$$\Gamma_2 \ : \ t^2 = J'(s) = 4s^3 - g_2 s - g_3$$

## Birational morphism of curves $\psi \ : \ \Gamma_2 \to \Gamma_1$

Induced by fractional-linear mapping $\hat{\psi}$ which maps zeros of the polynomial $J'$ and $\infty$ to the four zeros of the polynomial $P$.

# Two-valued group on $\mathbf{CP}^1$

There is a group structure on the cubic $\Gamma_2$. Together with its subgroup $\mathbf{Z}_2$, it defines the standrad two-valued group structure on $\mathbf{CP}^1$:

$$s_1 *_c s_2 = \left[ -s_1 - s_2 + \left( \frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2, -s_1 - s_2 + \left( \frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right],$$

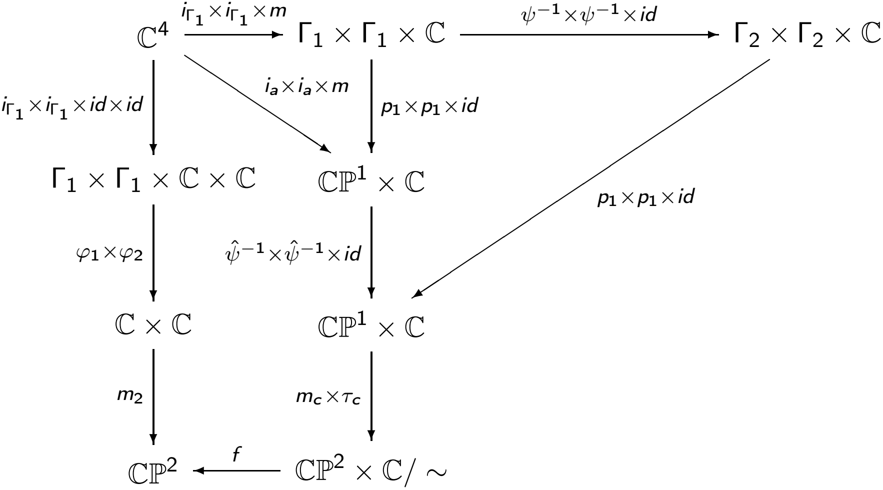
where $t_i = J'(s_i)$, $i = 1, 2$.

## Theorem [V. D. (2009)]

The general pencil equation after fractional-linear transformations

$$F(s, \hat{\psi}^{-1}(x_1), \hat{\psi}^{-1}(x_2)) = 0$$

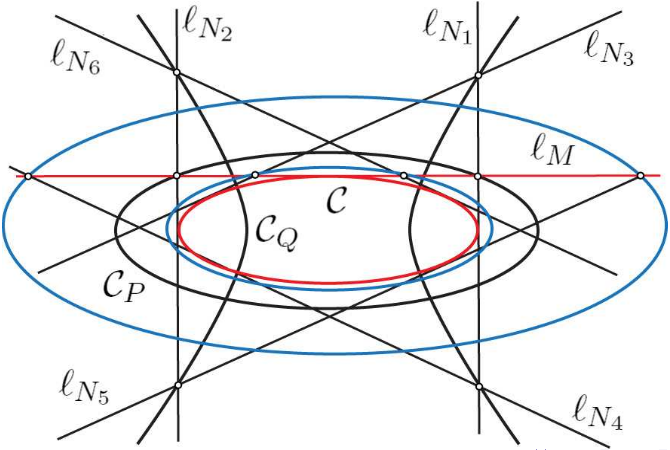
defines the two valued coset group structure $(\Gamma_2, \mathbb{Z}_2)$.


$\mathbb{C}^4$
$i_{\Gamma_1} \times i_{\Gamma_1} \times m$
$\Gamma_1 \times \Gamma_1 \times \mathbb{C}$
$\psi^{-1} \times \psi^{-1} \times id$
$\Gamma_2 \times \Gamma_2 \times \mathbb{C}$
$i_{\Gamma_1} \times i_{\Gamma_1} \times id \times id$
$i_a \times i_a \times m$
$p_1 \times p_1 \times id$
$\Gamma_1 \times \Gamma_1 \times \mathbb{C} \times \mathbb{C}$
$\mathbb{CP}^1 \times \mathbb{C}$
$p_1 \times p_1 \times id$
$\varphi_1 \times \varphi_2$
$\hat{\psi}^{-1} \times \hat{\psi}^{-1} \times id$
$\mathbb{C} \times \mathbb{C}$
$\mathbb{CP}^1 \times \mathbb{C}$
$m_2$
$m_c \times \tau_c$
$\mathbb{CP}^2$
$f$
$\mathbb{CP}^2 \times \mathbb{C}/\sim$

# Two-valued group $\mathbf{CP}^1$

**Theorem [V. D. (2009)]**

Associativity conditions for the group structure of the two-valued coset group $(\Gamma_2, \mathbb{Z}_2)$ and for its action on $\Gamma_1$ are equivalent to the great Poncelet theorem for a triangle.



# Experimental Math

## Other $n$-valued groups

$$p_3 = s_1^3 - 3^3 s_3$$

$$Dp_3 = y^2 x^2 (x - y)^2.$$

$$p_4 = s_1^4 - 2^3 s_1^2 s_2 + 2^4 s_2^2 - 2^7 s_1 s_3$$

$$Dp_4 = y^3 x^3 (x - y)^2 (y + 4x)^2 (4y + x)^2.$$

$$p_5 = s_1^5 - 5^4 s_1^2 s_3 + 5^5 s_2 s_3$$

$$Dp_5 = y^4 x^4 (x - y)^4 (-y^2 - 11xy + x^2)^2 (-y^2 + 11xy + x^2)^2.$$

# References

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