AUBRY-MATHER MEASURES IN THE NON CONVEX SETTING

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Abstract. The adjoint method introduced in [Eva] and [Tra] is used, to construct analogs to the Aubry-Mather measures for non convex Hamiltonians. More precisely, a general construction of probability measures, that in the convex setting agree with Mather measures, is provided. These measures may fail to be invariant under the Hamiltonian flow and a dissipation arises, which is described by a positive semi-definite matrix of Borel measures. However, in the important case of uniformly quasi-convex Hamiltonians the dissipation vanishes, and as a consequence the invariance is guaranteed.

Keywords:

1. Introduction

Let us consider a periodic Hamiltonian system whose energy is described by a smooth Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$. Here $\mathbb{T}^n$ denotes the n-dimensional torus, $n \in \mathbb{N}$. It is well known that the time evolution $t \mapsto (\mathbf{x}(t), \mathbf{p}(t))$ of the system is obtained by solving the Hamilton's ODE

$$\begin{cases} \dot{\mathbf{x}} = -D_p H(\mathbf{x}, \mathbf{p}), \\ \dot{\mathbf{p}} = D_x H(\mathbf{x}, \mathbf{p}). \end{cases} \tag{1.1}$$

Assume now that, for each $P \in \mathbb{R}^n$, there exists a constant $\overline{H}(P)$ and a periodic function $u(\cdot, P)$ solving the following time independent Hamilton-Jacobi equation

$$H(x, P + D_x u(x, P)) = \overline{H}(P). \tag{1.2}$$

Suppose, in addition, that both $u(x, P)$ and $\overline{H}(P)$ are smooth functions. Then, if the following relations

$$X = x + D_P u(x, P), \qquad p = P + D_x u(x, P), \tag{1.3}$$

define a smooth change of coordinates $X(x, p)$ and $P(x, p)$, the ODE (1.1) can be rewritten as

$$\begin{cases} \dot{\mathbf{X}} = -D_P \overline{H}(\mathbf{P}), \\ \dot{\mathbf{P}} = 0. \end{cases} \tag{1.4}$$

Since the solution of (1.4) is easily obtained, solving (1.1) is reduced to inverting the change of coordinates (1.3). Unfortunately, several difficulties arise.

Firstly, it is well known that the solutions of the nonlinear PDE (1.2) are not smooth in the general case. One can anyway solve (1.2) in a weaker sense, as made precise by the following theorem, due to Lions, Papanicolaou and Varadhan. For the convenience of the reader, we also recall the definition of viscosity solution.

Theorem 1.1 (See [LPV88])**.** *Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be smooth such that*

$$\lim_{|p|\to+\infty} H(x,p) = +\infty.$$

Then, for every $P \in \mathbb{R}^n$ there exists a unique $\overline{H}(P) \in \mathbb{R}$ such that the equation

$$H(x, P + D_x u(x,P)) = \overline{H}(P) \tag{1.5}$$

admits a $\mathbb{Z}^n$-periodic viscosity solution $u(\cdot, P) : \mathbb{T}^n \to \mathbb{R}$.

We call (1.5) the *cell problem.* It can be proved that all the viscosity solutions of the cell problem are Lipschitz continuous, with Lipschitz constants uniformly bounded in P.

Definition 1.2. We say that u is a *viscosity solution* of (1.5) if for each $v \in C^\infty(\mathbb{R}^n)$

- If $u - v$ has a local maximum at a point $x_0 \in \mathbb{R}^n$ then

$$H(x_0, P + D_x v(x_0)) \le \overline{H}(P);$$

- If $u - v$ has a local minimum at a point $x_0 \in \mathbb{R}^n$ then

$$H(x_0, P + D_x v(x_0)) \ge \overline{H}(P).$$

A second important issue is that the solution $u(\cdot, P)$ of (1.5) may not be unique, even modulo addition of constants. Indeed, a simple example is given by the Hamiltonian $H(x,p) = p\cdot(p - D\psi(x))$, where $\psi : \mathbb{T}^n \to \mathbb{R}$ is a smooth fixed function. In this case, for $P = 0$ and $\overline{H}(0) = 0$, the cell problem is

$$Du \cdot D(u - \psi) = 0,$$

which admits both $u \equiv 0$ and $u = \psi$ as solutions. Therefore, smoothness of $u(x,P)$ in P cannot be guaranteed.

Finally, even in the particular case in which both $u(x,P)$ and $\overline{H}(P)$ are smooth, relations (1.3) may not be invertible, or the functions $X(x,p)$ and $P(x,p)$ may not be smooth or globally defined.

Therefore, in order to understand the solutions of Hamilton's ODE (1.1) in the general case, it is very important to exploit the functions $\overline{H}(P)$ and $u(x,P)$, and to extract any possible information "encoded" in $\overline{H}(P)$ about the dynamics.

1.1. **Classical Results: the convex case.** Classically, the additional hypotheses required in literature on the Hamiltonian H are:

(i) $H(x,\cdot)$ is strictly convex;

(ii) $H(x,\cdot)$ is superlinear, i.e.

$$\lim_{|p|\to+\infty}\frac{H(x,p)}{|p|}=+\infty.$$

A typical example is the mechanical Hamiltonian

$$H(x,p)=\frac{|p|^2}{2}+V(x),$$

where V is a given smooth $\mathbb{Z}^n$-periodic function. Also, one restricts the attention to a particular class of trajectories of (1.1), the so-called one sided *absolute minimizers* of the action integral. More precisely, one first defines the Lagrangian $L:\mathbb{T}^n\times\mathbb{R}^n\to\mathbb{R}$ associated to H as the Legendre transform of H:

$$L(x,v):=H^*(x,v)=\sup_{p\in\mathbb{R}^n}\{-p\cdot v-H(x,p)\}\quad\text{for every } (x,v)\in\mathbb{T}^n\times\mathbb{R}^n. \tag{1.6}$$

Here the signs are set following the Optimal Control convention. Then, one looks for a Lipschitz curve $\mathbf{x}(\cdot)$ which minimizes the action integral, i.e. such that

$$\int_0^T L(\mathbf{x}(t),\dot{\mathbf{x}}(t))\,dt\le\int_0^T L(\mathbf{y}(t),\dot{\mathbf{y}}(t))\,dt \tag{1.7}$$

for each time $T>0$ and each Lipschitz curve $\mathbf{y}(\cdot)$ with $\mathbf{y}(0)=\mathbf{x}(0)$ and $\mathbf{y}(T)=\mathbf{x}(T)$.

Under fairly general conditions the minimizer exists, is smooth, and it satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\left[D_vL(\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]=D_xL(\mathbf{x}(t),\dot{\mathbf{x}}(t)),\quad t\in(0,+\infty). \tag{1.8}$$

It may be shown that if $\mathbf{x}(\cdot)$ solves (1.7) (and in turn (1.8)), then $(\mathbf{x}(\cdot),\mathbf{p}(\cdot))$ is a solution of (1.1), where $\mathbf{p}(\cdot):=-D_vL(\dot{\mathbf{x}}(\cdot),\mathbf{x}(\cdot)))$. This is a consequence of assumptions (i) and (ii), that in particular guarantee a one to one correspondence between Hamiltonian space and Lagrangian space coordinates, through the one to one map $\Phi:\mathbb{T}^n\times\mathbb{R}^n\to\mathbb{T}^n\times\mathbb{R}^n$ defined as

$$\Phi(x,v):=(x,-D_vL(x,v)). \tag{1.9}$$

There are several natural questions concerning trajectories $\mathbf{x}(\cdot)$ satisfying (1.7), in particular in what concerns ergodic averages, asymptotic behavior and so on. To address such questions it is common to consider the following related problem.

In 1991 John N. Mather (see [Mat91]) proposed a relaxed version of (1.7), by considering

$$\min_{\nu \in \mathcal{D}} \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x,v)\, d\nu(x,v), \tag{1.10}$$

where $\mathcal{D}$ is the class of probability measures in $\mathbb{T}^n \times \mathbb{R}^n$ that are invariant under the Euler-Lagrange flow. In Hamiltonian coordinates the property of invariance for a measure ν can be more conveniently written as:

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \{\phi, H\}\, d\mu(x,p) = 0, \qquad \forall\, \phi \in C^1_c(\mathbb{T}^n \times \mathbb{R}^n),$$

where $\mu = \Phi_\# \nu$ is the push-forward of the measure ν with respect to the map Φ, i.e. the measure μ such that

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,p)\, d\mu(x,p) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x, -D_v L(x,v))\, d\nu(x,v),$$

for every $\phi \in C(\mathbb{T}^n \times \mathbb{R}^n)$. Here the symbol $\{\cdot,\cdot\}$ stands for the Poisson bracket, that is

$$\{F, G\} := D_p F \cdot D_x G - D_x F \cdot D_p G, \qquad \forall\, F, G \in C^1(\mathbb{T}^n \times \mathbb{R}^n).$$

Denoting by $\mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ the class of probability measures on $\mathbb{T}^n \times \mathbb{R}^n$, we have

$$\mathcal{D} = \left\{ \nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n) : \int_{\mathbb{T}^n \times \mathbb{R}^n} \{\phi, H\}\, d\Phi_\# \nu(x,p) = 0, \qquad \forall\, \phi \in C^1_c(\mathbb{T}^n \times \mathbb{R}^n) \right\}. \tag{1.11}$$

The main disadvantage of problem (1.10) is that the set (1.11) where the minimization takes place depends on the Hamiltonian H and thus, in turn, on the integrand L. For this reason, Ricardo Mañe (see [Mn96]) considered the problem

$$\min_{\nu \in \mathcal{F}} \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x,v)\, d\nu(x,v), \tag{1.12}$$

where

$$\mathcal{F} := \left\{ \nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n) : \int_{\mathbb{T}^n \times \mathbb{R}^n} v \cdot D\psi(x)\, d\nu(x,v) = 0 \quad \text{for every } \psi \in C^1_c(\mathbb{T}^n) \right\}.$$

Measures belonging to $\mathcal{F}$ are also said to be *holonomic* measures. Notice that, in particular, to every trajectory $\mathbf{y}(\cdot)$ of the original problem (1.7) we can associate a measure $\nu_{\mathbf{y}(\cdot)} \in \mathcal{F}$. Indeed, for every $T > 0$ we can first define a measure $\nu_{T,\mathbf{y}(\cdot)} \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ by the relation

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,v)\, d\nu_{T,\mathbf{y}(\cdot)}(x,v) := \frac{1}{T} \int_0^T \phi(\mathbf{y}(t), \dot{\mathbf{y}}(t))\, dt \quad \text{for every } \phi \in C_c(\mathbb{T}^n \times \mathbb{R}^n).$$

Then, from the fact that

$$\operatorname{supp} \nu_{T,\mathbf{y}(\cdot)} \subset \mathbb{T}^n \times [-M, M], \quad \text{for every } T > 0, \qquad (M = \text{Lipschitz constant of } \mathbf{y}(\cdot))$$

we infer that there exists a sequence $T_j \to \infty$ and a measure $\nu_{\mathbf{y}(\cdot)} \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ such that $\nu_{T_j,\mathbf{y}(\cdot)} \overset{*}{\rightharpoonup} \mu_{\mathbf{y}(\cdot)}$ in the sense of measures, that is,

$$\lim_{j\to\infty} \frac{1}{T_j} \int_0^{T_j} \phi(\mathbf{y}(t), \dot{\mathbf{y}}(t))\, dt = \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x, v)\, d\nu_{\mathbf{y}(\cdot)}(x, v) \quad \text{for every } \phi \in C_c(\mathbb{T}^n \times \mathbb{R}^n). \tag{1.13}$$

Choosing $\phi(x, v) = v \cdot D\psi(x)$ in (1.13) it follows that $\nu_{\mathbf{y}(\cdot)} \in \mathcal{F}$, since

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} v \cdot D\psi(x)\, d\nu_{\mathbf{y}(\cdot)}(x, v) = \lim_{j\to\infty} \frac{1}{T_j} \int_0^{T_j} \dot{\mathbf{y}}(t) \cdot D\psi(\mathbf{y}(t))\, dt = \lim_{j\to\infty} \frac{\psi(\mathbf{y}(T_j)) - \psi(\mathbf{y}(0))}{T_j} = 0.$$

In principle, since $\mathcal{F}$ is much larger than the class of measures $\mathcal{D}$, we could expect the last problem not to have the same solution of (1.10). Instead, Mañe proved that every solution of (1.12) is also a minimizer of (1.10) and (1.7), where in (1.7) we identify Lipschitz trajectories with concentrated measures, as explained above.

A more general version of (1.12) consists in studying, for each $P \in \mathbb{R}^n$ fixed,

$$\min_{\nu\in\mathcal{F}} \int_{\mathbb{T}^n\times\mathbb{R}^n} (L(x, v) + P \cdot v)\; d\nu(x, v), \tag{1.14}$$

referred to as *Mather problem*. Any minimizer of (1.14) is said to be a *Mather measure*. An interesting connection between the Mather problem and the time independent Hamilton-Jacobi equation (1.5) is established by the identity:

$$-\overline{H}(P) = \min_{\nu\in\mathcal{F}} \int_{\mathbb{T}^n\times\mathbb{R}^n} (L(x, v) + P \cdot v)\; d\nu(x, v). \tag{1.15}$$

The following theorem gives a characterization of Mather measures in the convex case.

Theorem 1.3. *Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying (i) and (ii), and let $P \in \mathbb{R}^n$. Then, $\nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ is a solution of* (1.14) *if and only if:*

$$(a) \qquad \int_{\mathbb{T}^n\times\mathbb{R}^n} H(x, p)\, d\mu(x, p) = \overline{H}(P) = H(x, p) \qquad \mu\text{-a.e.};$$

$$(b) \qquad \int_{\mathbb{T}^n\times\mathbb{R}^n} (p - P) \cdot D_pH(x, p)\, d\mu(x, p) = 0;$$

$$(c) \qquad \int_{\mathbb{T}^n\times\mathbb{R}^n} D_pH(x, p) \cdot D_x\phi(x)\, d\mu(x, p) = 0, \qquad \forall\, \phi \in C^1(\mathbb{T}^n),$$

where $\mu = \Phi_{\#}\nu$ and $\overline{H}(P)$ is defined by Theorem 1.1.

Before proving Theorem 1.3 we state the following proposition. For the proof, see [Mn96].

Proposition 1.4. *Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying (i) and (ii). Let $P \in \mathbb{R}^n$, let $\nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ be a minimizer of* (1.14) *and set $\mu = \Phi_{\#}\nu$. Then,*

(1) *μ is invariant under the Hamiltonian dynamics, i.e.*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \{\phi, H\}\, d\mu(x,p) = 0 \qquad \forall\, \phi \in C^1_c(\mathbb{T}^n\times\mathbb{R}^n);$$

(2) *μ is supported on the graph*

$$\Sigma := \{(x,p) \in \mathbb{T}^n\times\mathbb{R}^n : p = P + D_x u(x)\},$$

where u is any viscosity solution of (1.5).

We observe that property (2), also known as the *graph theorem*, is a highly nontrivial result. Indeed, by using hypothesis (ii) one can show that any solution $u(\cdot,P)$ of (1.5) is Lipschitz continuous, but higher regularity cannot be expected in the general case.

Proof of Theorem 1.3. To simplify, we will assume $P = 0$.

Let ν be a minimizer of (1.14). By the previous proposition, we know that properties (1) and (2) hold; let us prove that $\mu = \Phi_{\#}\nu$ satisfies (a)–(c). By (1.15), we have

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,v)\, d\nu(x,v) = -\overline{H}(0).$$

Furthermore, because of (2)

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} H(x,p)\, d\mu(x,p) = \overline{H}(0),$$

that is, (a). Since $H(x,p) = -L(x, D_pH(x,p)) - p\cdot D_pH(x,p)$, this implies that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} p\cdot D_pH(x,p)\, d\mu(x,p) = 0,$$

and so (b) holds. Finally, (c) follows directly from the fact that $\nu \in \mathcal{F}$.

Let now $\mu \in \mathcal{P}(\mathbb{T}^n\times\mathbb{R}^n)$ satisfy (a)–(c), and let us show that $\nu = (\Phi^{-1})_{\#}\mu$ is a minimizer of (1.14). First of all, observe that $\nu \in \mathcal{F}$. Indeed, by using (c) for every $\psi \in C^1(\mathbb{T}^n)$

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} v\cdot D\psi(x)\, d\nu(x,v) = \int_{\mathbb{T}^n\times\mathbb{R}^n} D_pH(x,p)\cdot D\psi(x)\, d\mu(x,p) = 0.$$

Let now prove that ν is a minimizer.

Integrating equality $H(x,p) = -L(x, D_pH(x,p)) - p\cdot D_pH(x,p)$ with respect to μ, and using (a) and (c) we have

$$\begin{aligned}
\overline{H}(0) &= \int_{\mathbb{T}^n\times\mathbb{R}^n} H(x,p)\, d\mu(x,p) \\
&= -\int_{\mathbb{T}^n\times\mathbb{R}^n} L(x, D_pH(x,p))\, d\mu(x,p) - \int_{\mathbb{T}^n\times\mathbb{R}^n} p\cdot D_pH(x,p)\, d\mu(x,p) \\
&= -\int_{\mathbb{T}^n\times\mathbb{R}^n} L(x, D_pH(x,p))\, d\mu(x,p) = -\int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,v)\, d\nu(x,v).
\end{aligned}$$

By (1.15), ν is a minimizer of (1.14).

□

1.2. **The Non Convex Case.** The main goal of this paper is to use the techniques of [Eva] and [Tra], to construct Mather measures under fairly general hypotheses, when the variational construction just described cannot be used. Indeed, when (i) and (ii) are satisfied H coincides with the Legendre transform of L, that is, identity $H = H^{**}$ holds. Moreover, L turns out to be convex and superlinear as well, and relation (1.9) defines a smooth diffeomorphism, that allows to pass from Hamiltonian to Lagrangian coordinates.

First of all, we extend the definition of Mather measure to the non convex setting, without making use of the Lagrangian formulation.

Definition 1.5. We say that a measure $\mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ is a *Mather measure* if there exists $P \in \mathbb{R}^n$ such that properties (a)–(c) are satisfied.

The results exposed in the previous subsection show that, modulo the push-forward operation, this definition is equivalent to the usual one in literature (see e.g. [Fat], [Mn96], [Mat91]). We would like now to answer the following natural questions:

- **Question 1:** Does a Mather measure exist?
- **Question 2:** Let μ be a Mather measure. Are properties (1) and (2) satisfied?

We just showed that in the convex setting both questions have affirmative answers. Before addressing these issues, let us make some hypotheses on the Hamiltonian H. We remark that without any coercivity assumption (i.e. without any condition similar to (ii)), there are no a priori bounds for the modulus of continuity of periodic solutions of (1.5). Indeed, for $n = 2$ consider the Hamiltonian

$$H(x,p) = p_1^2 - p_2^2 \qquad \forall\, p = (p_1, p_2) \in \mathbb{R}^2.$$

In this case, equation (1.5) for $P = 0$ and $\overline{H}(P) = 0$ becomes

$$u_x^2 - u_y^2 = 0. \tag{1.16}$$

Then, for every choice of $f : \mathbb{R} \to \mathbb{R}$ of class C^1, the function $u(x,y) = f(x-y)$ is a solution of (1.16). Clearly, there are no uniform Lipschitz bounds for the family of all such functions u. Throughout all the paper, we will assume that

(H1) H is smooth;

(H2) $H(\cdot,p)$ is $\mathbb{Z}^n$-periodic for every $p \in \mathbb{R}^n$;

(H3) For every $x \in \mathbb{T}^n$

$$\lim_{|p|\to+\infty} \left(\frac{1}{2}|H(x,p)|^2 + D_x H(x,p)\cdot p \right) = +\infty.$$

Notice that if we have hypothesis (ii) of the previous subsection and a bound on $D_xH(x,p)$, e.g. $|D_xH(x,p)| \le C(1+|p|)$, then we obviously have (H3).

First we consider, for every $\varepsilon > 0$, a regularized version of (1.5), showing existence and uniqueness of a constant $\overline{H}^{\varepsilon}(P)$ such that

$$-\frac{\varepsilon^2}{2}\Delta u^{\varepsilon}(x) + H(x, P + Du^{\varepsilon}(x)) = \overline{H}^{\varepsilon}(P) \tag{1.17}$$

admits a $\mathbb{Z}^n$-periodic viscosity solution (see Theorem 2.1) . Thanks to (H3), we can establish a uniform bound on $\|Du^{\epsilon}\|_{L^\infty}$, and prove that, up to subsequences, $\overline{H}^{\varepsilon}(P) \to \overline{H}(P)$ and $u^{\varepsilon}(\cdot, P)$ converge uniformly to $u(\cdot, P)$ as $\varepsilon \to 0$, where $\overline{H}(P)$ and $u(\cdot, P)$ solve equation (1.5). Then, for every $\varepsilon > 0$ we define the perturbed Hamilton SDE (see Section 3) as

$$\begin{cases} d\mathbf{x}^{\varepsilon} = -D_pH(\mathbf{x}^{\varepsilon}, \mathbf{p}^{\varepsilon})\,dt + \varepsilon\,dw_t, \\ d\mathbf{p}^{\varepsilon} = D_xH(\mathbf{x}^{\varepsilon}, \mathbf{p}^{\varepsilon})\,dt + \varepsilon D^2 u^{\varepsilon}\,dw_t, \end{cases} \tag{1.18}$$

where w_t is a n-dimensional Brownian motion. However, it is not necessary to use the stochastic approach, since our techniques can also be introduced in a purely PDE way (see Section 3.3).

In the second step, in analogy to what is done in the convex setting, we encode the long-time behavior of the solutions $t \mapsto (\mathbf{x}^{\varepsilon}(t), \mathbf{p}^{\varepsilon}(t))$ of (1.18) into a family of probability measures $\{\mu^{\varepsilon}\}_{\varepsilon>0}$, defined by

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,p)\,d\mu^{\varepsilon}(x,p) := \lim_{T_j\to\infty} \frac{1}{T_j}\,E\left[\int_0^{T_j} \phi(\mathbf{x}^{\varepsilon}(t), \mathbf{p}^{\varepsilon}(t))\,dt\right] \qquad \text{for every } \phi \in C_c(\mathbb{T}^n \times \mathbb{R}^n),$$

where with $E[\cdot]$ we denote the expected value and the limit is taken along appropriate subsequences $\{T_j\}_{j\in\mathbb{N}}$ (see Section 3.1).

Using the techniques developed in [Eva], we are able to provide some bounds on the derivatives of the functions u^{ε}. More precisely, defining $\theta_{\mu^{\varepsilon}}$ as the projection on the torus $\mathbb{T}^n$ of the measure μ^{ε} (see Section 3.2), we give estimates on the $(L^2, d\theta_{\mu^{\varepsilon}})$-norm of the second and third derivatives of u^{ε}, uniformly w.r.t. ε (see Proposition 4.1).

In this way, we show that there exist a Mather measure μ and a nonnegative, symmetric $n \times n$ matrix of Borel measures $(m_{kj})_{k,j=1,\dots,n}$ such that μ^{ε} converges to μ up to subsequences and

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \{\phi, H\}\,d\mu + \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi_{p_k p_j}\,dm_{kj} = 0, \qquad \forall\,\phi \in C_c^2(\mathbb{T}^n \times \mathbb{R}^n), \tag{1.19}$$

with sum understood over repeated indices (see Theorem 5.1). As in [Eva], we call m_{kj} the *dissipation measures*. Relation (1.19) is the key point of our work, since it immediately shows the differences with the convex case. Indeed, the Mather measure μ is invariant under the Hamiltonian flow if and only the dissipation measures m_{kj} vanish. When $H(x,\cdot)$ is convex, this is guaranteed by an improved version of the estimates on the second derivatives of u^ε (see Proposition 4.1, estimate (4.4)). We sketch in Section 10 a one dimensional case showing that, in general, the dissipation measures $(m_{kj})_{k,j=1,\dots,n}$ do not disappear. Nevertheless, we are able to provide some interesting examples of non-convex Hamiltonians (see Section 9), for which both properties (1) and (2) are satisfied, e.g. strictly quasi-convex Hamiltonians (see Section 9.7).

2. Elliptic regularization of the cell problem

The elliptic regularization of (1.5), or the vanishing viscosity method is a well known tool to study viscosity solutions. In the context of Mather measures this procedure was introduced by Gomes in [Gom02], see also [Ana04], [AIPSM05], [ISM05].

We start by quoting a classical result concerning an elliptic regularization of equation (1.5).

Theorem 2.1. *For every $\varepsilon > 0$ and every $P \in \mathbb{R}^n$, there exists a unique number $\overline{H}^\varepsilon(P) \in \mathbb{R}$ such that the equation*

$$-\frac{\varepsilon^2}{2}\Delta u^\varepsilon(x) + H(x, P + Du^\varepsilon(x)) = \overline{H}^\varepsilon(P) \tag{2.1}$$

admits a unique (up to constants) $\mathbb{Z}^n$-periodic viscosity solution. Moreover, for every $P \in \mathbb{R}^n$

$$\lim_{\varepsilon\to 0^+} \overline{H}^\varepsilon(P) = \overline{H}(P),$$

where $\overline{H}(P)$ is given by Theorem 1.1. In addition, we have

$$u^\varepsilon \to u \qquad\qquad \text{uniformly,}$$

where u is a $\mathbb{Z}^n$-periodic viscosity solution of (1.5).

Definition 2.2. Let $\varepsilon > 0$ and $P \in \mathbb{R}^n$. The *linearized operator* $L^{\varepsilon,P} : C^2(\mathbb{T}^n) \to C(\mathbb{T}^n)$ associated to equation (2.1) is defined as

$$L^{\varepsilon,P}v(x) := -\frac{\varepsilon^2}{2}\Delta v(x) + D_pH(x, P + Du^\varepsilon(x)) \cdot Dv(x),$$

for every $v \in C^2(\mathbb{T}^n)$.

We call (2.1) the *stochastic cell problem.*

Sketch of the Proof. We mimic the proofs in [LPV88] and [Gom02]. Let's consider the following problem

$$\lambda v^\lambda + H(x, P + Dv^\lambda) = \frac{\epsilon^2}{2}\Delta v^\lambda.$$

The above equation has a unique smooth solution v^λ in $\mathbb{R}^n$ which is $\mathbb{Z}^n$-periodic.

What we need to prove now is $||\lambda v^\lambda||_{L^\infty}, ||Dv^\lambda||_{L^\infty} \le C$ for some constant $C > 0$.

By using the Maximum principle with $\varphi = 0$ as a test function, we easily get $||\lambda v^\lambda||_{L^\infty} \leq C$. We now only need to deal with $||Dv^\lambda||_{L^\infty}$. Let $w^\lambda = \dfrac{|Dv^\lambda|^2}{2}$ then we have

$$2\lambda w^\lambda + D_pH.Dw^\lambda + D_xH.Dv^\lambda = \frac{\epsilon^2}{2}\Delta w^\lambda - \frac{\epsilon^2}{2}|D^2v^\lambda|^2.$$

Notice that for $\epsilon < 1$ we have

$$\frac{\epsilon^2}{2}|D^2v^\lambda|^2 \geq \frac{\epsilon^4}{4}|\Delta v^\lambda|^2 = (\lambda v^\lambda + H)^2 \geq \frac{1}{2}H^2 - C.$$

Therefore,

$$2\lambda w^\lambda + D_pH.Dw^\lambda + D_xH.Dv^\lambda + \frac{1}{2}H^2 - C \leq \frac{\epsilon^2}{2}\Delta w^\lambda.$$

At $x_1 \in \mathbb{T}^n$ where $w^\lambda(x_1) = \max_{\mathbb{T}^n} w^\lambda$

$$2\lambda w^\lambda(x_1) + D_xH.Dv^\lambda(x_1) + \frac{1}{2}H^2 \leq C.$$

By condition (iii) and $w^\lambda(x_1) \geq 0$, w^λ is bounded independently of λ, ϵ and we conclude the proof. □

Remark 2.3. The classical theory (see Lions [Lio82]) ensures that the functions u^ε are C^∞. In addition, the functions u^ε are Lipschitz, with a Lipschitz constant independent of ε by the above theorem.

3. Stochastic dynamics

We now introduce a stochastic dynamics associated with the stochastic cell problem (2.1). This will be a perturbation to the Hamiltonian dynamics (1.1), which describes the trajectory in the phase space of a classical mechanical system.

Let (Ω, σ, P) be a probability space, and let w_t be a n-dimensional Brownian motion on Ω. Let $\varepsilon > 0$, and let u^ε be a $\mathbb{Z}^n$-periodic solution of (2.1). To simplify, we set $P = 0$. Consider now the solution $\mathbf{x}^\varepsilon(t)$ of

$$\begin{cases} d\mathbf{x}^\varepsilon = -D_pH(Du^\varepsilon(\mathbf{x}^\varepsilon), \mathbf{x}^\varepsilon)\,dt + \varepsilon\,dw_t, \\ \mathbf{x}^\varepsilon(0) = \overline{x}, \end{cases} \tag{3.1}$$

with $\overline{x} \in \mathbb{T}^n$ arbitrary. Accordingly, the momentum variable is defined as

$$\mathbf{p}^\varepsilon(t) = Du^\varepsilon(\mathbf{x}^\varepsilon(t)).$$

Suppose $\mathbf{z}$ is a solution to the SDE:

$$d\mathbf{z}_i = a_i\,dt + b_{ij}\,w_t^j \qquad\qquad i = 1, \ldots, n,$$

with a_i and b_{ij} bounded and progressively measurable processes. Let $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a smooth function. Then, $\varphi(\mathbf{z}, t)$ satisfies the *Itô formula*:

$$d\varphi = \varphi_{z_i}\, dz_i + \left(\varphi_t + \frac{1}{2} b_{ij} b_{jk} \varphi_{z_i z_k}\right) dt. \tag{3.2}$$

An integrated version of the Itô formula is the *Dynkin's formula*:

$$E\left[\phi(\mathbf{z}(T)) - \phi(\mathbf{z}(0))\right] = E\left[\int_0^T \left(a_i D_{z_i}\phi(\mathbf{z}(t)) + \frac{1}{2} b_{ij} b_{jk} D^2_{z_i z_k}\phi(\mathbf{z}(t))\right) dt\right].$$

Here and always in the sequel, we use Einstein convention for repeated indices in a sum. In the present situation, we have

$$a_i = -D_{p_i} H(Du^\varepsilon, \mathbf{x}^\varepsilon), \qquad b_{ij} = \varepsilon \delta_{ij}.$$

Hence, recalling (3.1) and (3.2)

$$\begin{aligned} dp_i &= u^\varepsilon_{x_i x_j}\, dx^\varepsilon_j + \frac{\varepsilon^2}{2} \Delta(u^\varepsilon_{x_i})\, dt = -L^{\varepsilon,P} u^\varepsilon_{x_i} dt + \varepsilon u^\varepsilon_{x_i x_j}\, dw^j_t \\ &= D_{x_i} H\, dt + \varepsilon u^\varepsilon_{x_i x_j}\, dw^j_t, \end{aligned}$$

where in the last equality we used identity (4.8). Thus, $(\mathbf{x}^\varepsilon, \mathbf{p}^\varepsilon)$ satisfies the following stochastic version of the Hamiltonian dynamics (1.1):

$$\begin{cases} d\mathbf{x}^\varepsilon = -D_p H(\mathbf{x}^\varepsilon, \mathbf{p}^\varepsilon)\, dt + \varepsilon\, dw_t, \\ d\mathbf{p}^\varepsilon = D_x H(\mathbf{x}^\varepsilon, \mathbf{p}^\varepsilon)\, dt + \varepsilon D^2 u^\varepsilon\, dw_t. \end{cases} \tag{3.3}$$

We are now going to study the behavior of the solutions u^ε of equation (2.1) along the trajectory $\mathbf{x}^\varepsilon(t)$. Thanks to the Itô formula and relations (3.3) and (2.1):

$$\begin{aligned} du^\varepsilon(\mathbf{x}^\varepsilon(t)) &= Du^\varepsilon d\mathbf{x}^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon\, dt = -L^{\varepsilon,P} u^\varepsilon dt + \varepsilon Du^\varepsilon\, dw_t \\ &= \left(H - \overline{H}^\varepsilon(P) - D_x u^\varepsilon D_p H\right) dt + \varepsilon Du^\varepsilon\, dw_t. \end{aligned} \tag{3.4}$$

Using Dynkin's formula in (3.4) we obtain

$$E\big(u^\varepsilon(\mathbf{x}^\varepsilon(T)) - u^\varepsilon(\mathbf{x}^\varepsilon(0))\big) = E\left[\int_0^T \left(H - \overline{H}^\varepsilon(P) - D_x u^\varepsilon D_p H\right) dt\right].$$

In the convex case, since the Lagrangian L is related with the Hamiltonian by the relation

$$L = p \cdot D_p H - H,$$

we have

$$u^\varepsilon(\mathbf{x}^\varepsilon(0)) = E\left[\int_0^T \left(L + \overline{H}^\varepsilon(P)\right) dt + u^\varepsilon(\mathbf{x}^\varepsilon(T))\right].$$

3.1. **Phase space measures.** We will encode the asymptotic behaviour of the trajectories by considering ergodic averages. More precisely, we associate to every trajectory $(\mathbf{x}^\varepsilon(\cdot), \mathbf{p}^\varepsilon(\cdot))$ of (3.3) a probability measure $\mu^\varepsilon \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ defined by

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,p)\, d\mu^\varepsilon(x,p) := \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \phi(\mathbf{x}^\varepsilon(t), \mathbf{p}^\varepsilon(t))\, dt\right],$$

for every $\phi \in C_c(\mathbb{T}^n \times \mathbb{R}^n)$. In the expression above, the definition makes sense provided the limit is taken over an appropriate subsequence. Moreover, no uniqueness is asserted, since by choosing a different subsequence one can in principle obtain a different limit measure μ^ε. Then, using Dynkin's formula we have, for every $\phi \in C_c^2(\mathbb{T}^n \times \mathbb{R}^n)$,

$$\begin{aligned} E\left[\phi(\mathbf{x}^\varepsilon(T), \mathbf{p}^\varepsilon(T)) - \phi(\mathbf{x}^\varepsilon(0), \mathbf{p}^\varepsilon(0))\right] &= E\left[\int_0^T \Big(D_p\phi \cdot D_x H - D_x\phi \cdot D_p H\Big)\, dt\right] \\ &+ E\left[\int_0^T \Big(\frac{\varepsilon^2}{2}\phi_{x_i x_i} + \varepsilon^2 u^\varepsilon_{x_i x_j}\phi_{x_i p_j} + \frac{\varepsilon^2}{2} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_j}\phi_{p_k p_j}\Big)\, dt\right]. \end{aligned} \tag{3.5}$$

Dividing last relation by T and passing to the limit as $T \to +\infty$ (along a suitable subsequence) we obtain

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \{\phi, H\}\, d\mu^\varepsilon + \int_{\mathbb{T}^n \times \mathbb{R}^n} \left[\frac{\varepsilon^2}{2}\phi_{x_i x_i} + \varepsilon^2 u^\varepsilon_{x_i x_j}\phi_{x_i p_j} + \frac{\varepsilon^2}{2} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_j}\phi_{p_k p_j}\right] d\mu^\varepsilon = 0. \tag{3.6}$$

3.2. **Projected measure.** To every probability measure $\mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$, we associate the *projected measure* $\theta_\mu \in \mathcal{P}(\mathbb{T}^n)$ in the following way:

$$\int_{\mathbb{T}^n} \varphi(x)\, d\theta_\mu(x) := \int_{\mathbb{T}^n \times \mathbb{R}^n} \varphi(x)\, d\mu(x,p), \qquad \forall\, \varphi \in C(\mathbb{T}^n).$$

Using test functions that do not depend on the variable p in the previous definition we conclude from identity (3.6) that

$$\int_{\mathbb{T}^n} D_p H \cdot D_x\varphi\; d\theta_{\mu^\varepsilon} = \frac{\varepsilon^2}{2}\int_{\mathbb{T}^n \times \mathbb{R}^n} \Delta\varphi\, d\theta_{\mu^\varepsilon}, \qquad \forall\, \varphi \in C^2(\mathbb{T}^n). \tag{3.7}$$

3.3. **PDE Approach.** The measures μ^ε and θ_{μ^ε} can be defined also by using standard PDE methods from (3.7). Indeed, given u^ε we can consider the PDE

$$\frac{\varepsilon^2}{2}\Delta\theta^\varepsilon + \operatorname{div}\left(D_p H(x, P + D_x u^\varepsilon)\, \theta^\varepsilon\right) = 0,$$

which admits a unique non-negative solution θ^ε with

$$\int_{\mathbb{T}^n} d\theta^\varepsilon(x) = 1,$$

since it is not hard to see that 0 is the principal eigenvalue of the elliptic operator $-\frac{\varepsilon^2}{2}\Delta v - \operatorname{div}(D_p H(x, P + D_x u^\varepsilon)\, v)$ in $\mathbb{T}^n$.

Then μ^ε can be defined as the unique measure such that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x,p)\, d\mu^\varepsilon(x,p) = \int_{\mathbb{T}^n} \psi(x, P + D_x u^\varepsilon(x))\, d\theta^\varepsilon(x),$$

for every $\psi \in C(\mathbb{T}^n \times \mathbb{R}^n)$. Then, identity (3.6) requires some work but can also be proved in a purely analytic way.

4. Uniform estimates

In this section we derive several estimates that will be useful when passing to the limit as $\varepsilon \to 0$. We will use here the same techniques as in [Eva] and [Tra].

Proposition 4.1. *We have the following estimates:*

$$\varepsilon^2 \int_{\mathbb{T}^n} |D^2 u^\varepsilon|^2\, d\theta_{\mu^\varepsilon} \le C, \tag{4.1}$$

$$\varepsilon^2 \int_{\mathbb{T}^n} |D^2_{Px} u^\varepsilon|^2\, d\theta_{\mu^\varepsilon} \le \int_{\mathbb{T}^n} |D_P u^\varepsilon|^2\, d\theta_{\mu^\varepsilon} + \int_{\mathbb{T}^n} |D_p H - D_P \overline{H}^\varepsilon|^2\, d\theta_{\mu^\varepsilon}, \tag{4.2}$$

$$\int_{\mathbb{T}^n} |D u^\varepsilon_{x_i x_i}|^2\, d\theta_{\mu^\varepsilon} \le \frac{C}{\varepsilon^2}\left(1 + \int_{\mathbb{T}^n} |D^2 u^\varepsilon|^3\, d\theta_{\mu^\varepsilon}\right), \qquad i = 1, \dots, n. \tag{4.3}$$

In addition, if H is uniformly convex in p, inequality (4.1) *can be improved to:*

$$\int_{\mathbb{T}^n} |D^2 u^\varepsilon|^2\, d\theta_{\mu^\varepsilon} \le C. \tag{4.4}$$

Here C denotes a positive constant independent of ε.

Remark 4.2. Estimate (4.4) was already proven in [Eva] and [Tra].

To prove the proposition we first need an auxiliary lemma. In the following, we denote by β either a direction in $\mathbb{R}^n$ (i.e. $\beta \in \mathbb{R}^n$ with $|\beta| = 1$), or a parameter (e.g. $\beta = P_i$ for some $i \in \{1, \dots, n\}$). When $\beta = P_i$ for some $i \in \{1, \dots, n\}$ the symbols H_β and $H_{\beta\beta}$ have to be understood as H_{p_i} and $H_{p_i p_i}$, respectively.

Lemma 4.3. *We have*

$$\varepsilon^2 \int_{\mathbb{T}^n} |D u^\varepsilon_\beta|^2\, d\theta_{\mu^\varepsilon} = 2 \int_{\mathbb{T}^n} u^\varepsilon_\beta (\overline{H}^\varepsilon_\beta - H_\beta)\, d\theta_{\mu^\varepsilon}, \tag{4.5}$$

$$\int_{\mathbb{T}^n} (\overline{H}^\varepsilon_{\beta\beta} - H_{\beta\beta} - 2 D_p H_\beta \cdot D u^\varepsilon_\beta - D^2_{pp} H D u^\varepsilon_\beta \cdot D u^\varepsilon_\beta)\, d\theta_{\mu^\varepsilon} = 0, \tag{4.6}$$

$$\varepsilon^2 \int_{\mathbb{T}^n} |D u^\varepsilon_{\beta\beta}|^2\, d\theta_{\mu^\varepsilon} = 2 \int_{\mathbb{T}^n} u^\varepsilon_{\beta\beta} (\overline{H}^\varepsilon_{\beta\beta} - H_{\beta\beta} - 2 D_p H_\beta \cdot D u^\varepsilon_\beta - D^2_{pp} H : D u^\varepsilon_\beta \otimes D u^\varepsilon_\beta\, d\theta_{\mu^\varepsilon}. \tag{4.7}$$

Proof. By differentiating equation (2.1) with respect to β and recalling Definition 2.2 we get

$$L^{\varepsilon,P} u^\varepsilon_\beta = \overline{H}^\varepsilon_\beta - H_\beta, \tag{4.8}$$

so that

$$\frac{1}{2}L^{\varepsilon,P}(|u^\varepsilon_\beta|^2) = u^\varepsilon_\beta L^{\varepsilon,P}u^\varepsilon_\beta - \frac{\varepsilon^2}{2}|Du^\varepsilon_\beta|^2 = u^\varepsilon_\beta(\overline{H}^\varepsilon_\beta - H_\beta) - \frac{\varepsilon^2}{2}|Du^\varepsilon_\beta|^2.$$

Integrating w.r.t. θ_{μ^ε} and recalling (3.7) we get (4.5).

To prove (4.6), we differentiate (4.8) w.r.t. β obtaining

$$L^{\varepsilon,P}u^\varepsilon_{\beta\beta} = \overline{H}^\varepsilon_{\beta\beta} - H_{\beta\beta} - 2D_pH_\beta \cdot Du^\varepsilon_\beta - D^2_{pp}H : Du^\varepsilon_\beta \otimes Du^\varepsilon_\beta. \tag{4.9}$$

Integrating w.r.t. θ_{μ^ε} and recalling (3.7) equality (4.6) follows. Finally, using (4.9)

$$\begin{aligned}\frac{1}{2}L^{\varepsilon,P}(|u^\varepsilon_{\beta\beta}|^2) &= u^\varepsilon_{\beta\beta}L^{\varepsilon,P}u^\varepsilon_{\beta\beta} - \frac{\varepsilon^2}{2}|Du^\varepsilon_{\beta\beta}|^2\\ &= u^\varepsilon_{\beta\beta}(\overline{H}^\varepsilon_{\beta\beta} - H_{\beta\beta} - 2D_pH_\beta \cdot Du^\varepsilon_\beta - D^2_{pp}H : Du^\varepsilon_\beta \otimes Du^\varepsilon_\beta) - \frac{\varepsilon^2}{2}|Du^\varepsilon_{\beta\beta}|^2.\end{aligned}$$

Once again, we integrating w.r.t. θ_{μ^ε} and use (3.7) to get (4.7). □

We can now proceed to the proof of Proposition 4.1.

Proof of Proposition 4.1. Thanks to estimate (4.5) we have

$$\varepsilon^2\int_{\mathbb{T}^n}|Du^\varepsilon_{x_i}|^2\,d\theta_{\mu^\varepsilon} = -2\int_{\mathbb{T}^n}Du^\varepsilon\cdot DH\,d\theta_{\mu^\varepsilon}, \qquad i=1,\ldots,n.$$

Since the functions u^ε are uniformly (in ε) Lipschitz (see Remark 2.3) (4.1) follows. Relation (4.2) follows by adding up (4.5) with $\beta = P_1, P_2, \ldots, P_n$ which yields

$$\frac{\varepsilon^2}{2}\int_{\mathbb{T}^n}|D^2_{Px}u^\varepsilon|^2\,d\theta_{\mu^\varepsilon} = \int_{\mathbb{T}^n}D_Pu^\varepsilon\cdot\left[D_pH - D_P\overline{H}^\varepsilon\right]\,d\theta_{\mu^\varepsilon}.$$

Let us show (4.3). Thanks to (4.7)

$$\begin{aligned}&\varepsilon^2\int_{\mathbb{T}^n}|Du^\varepsilon_{x_ix_i}|^2\,d\theta_{\mu^\varepsilon}\\ &\qquad = -2\int_{\mathbb{T}^n}u^\varepsilon_{x_ix_i}(H_{x_ix_i} + 2D_pH_{x_i}\cdot Du^\varepsilon_{x_i} + D^2_{pp}H : Du^\varepsilon_{x_i}\otimes Du^\varepsilon_{x_i})\,d\theta_{\mu^\varepsilon}.\end{aligned}$$

Since the functions u^ε are uniformly Lipschitz, we have

$$|H_{x_ix_i}|,\ |D_pH_{x_i}|,\ |D^2_{pp}H| \le C, \quad \text{ on the support of } \mu^\varepsilon.$$

Hence,

$$\begin{aligned}\varepsilon^2\int_{\mathbb{T}^n}|Du^\varepsilon_{x_ix_i}|^2\,d\theta_{\mu^\varepsilon} &\le C\left[\int_{\mathbb{T}^n}|D^2u^\varepsilon|\,d\theta_{\mu^\varepsilon} + \int_{\mathbb{T}^n}|D^2u^\varepsilon|^2\,d\theta_{\mu^\varepsilon} + \int_{\mathbb{T}^n}|D^2u^\varepsilon|^3\,d\theta_{\mu^\varepsilon}\right]\\ &\le C\left(1 + \int_{\mathbb{T}^n}|D^2u^\varepsilon|^3\,d\theta_{\mu^\varepsilon}\right).\end{aligned}$$

Finally, assume that H is uniformly convex and let us show (4.4). Thanks to (4.6) for every $i = 1, \dots, n$

$$
\begin{aligned}
0 &= \int_{\mathbb{T}^n} (H_{x_i x_i} + 2D_p H_{x_i} \cdot Du^\varepsilon_{x_i} + D^2_{pp} H Du^\varepsilon_{x_i} \cdot Du^\varepsilon_{x_i})\, d\theta_{\mu^\varepsilon} \\
&\geq \int_{\mathbb{T}^n} \left(H_{x_i x_i} + 2D_{p_i} H_{x_i} \cdot Du^\varepsilon_{x_i}\right)\, d\theta_{\mu^\varepsilon} + \alpha \|Du^\varepsilon_{x_i}\|^2_{L^2(\mathbb{T}^n; d\theta_{\mu^\varepsilon})},
\end{aligned}
$$

for some $\alpha > 0$. Thus, using Cauchy's and Young's inequality, for every $\eta \in \mathbb{R}$

$$
\begin{aligned}
\alpha \|Du^\varepsilon_{x_i}\|^2_{L^2(\mathbb{T}^n; d\theta_{\mu^\varepsilon})} &\leq - \int_{\mathbb{T}^n} H_{x_i x_i}\, d\theta_{\mu^\varepsilon} + 2\|D_p H_{x_i}\|_{L^2(\mathbb{T}^n; d\theta_{\mu^\varepsilon})} \|Du^\varepsilon_{x_i}\|_{L^2(\mathbb{T}^n; d\theta_{\mu^\varepsilon})} \\
&\leq - \int_{\mathbb{T}^n} H_{x_i x_i}\, d\theta_{\mu^\varepsilon} + \frac{1}{\eta^2} \|D_p H_{x_i}\|^2_{L^2(\mathbb{T}^n; d\theta_{\mu^\varepsilon})} + \eta^2 \|Du^\varepsilon_{x_i}\|^2_{L^2(\mathbb{T}^n; d\theta_{\mu^\varepsilon})}.
\end{aligned}
$$

Finally,

$$
(\alpha - \eta^2) \|Du^\varepsilon_{x_i}\|^2_{L^2(\mathbb{T}^n; d\theta_{\mu^\varepsilon})} \leq - \int_{\mathbb{T}^n} H_{x_i x_i}\, d\theta_{\mu^\varepsilon} + \frac{1}{\eta^2} \|D_p H_{x_i}\|^2_{L^2(\mathbb{T}^n; d\theta_{\mu^\varepsilon})}.
$$

Choosing $\eta^2 < \alpha$ we conclude the proof. ☐

5. Existence of Mather measures and dissipation measures

We now look at the asymptotic behavior of the measures μ^ε as $\varepsilon \to 0$, proving existence of Mather measures. The main result of the section is the following.

Theorem 5.1. *Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying condition (iii), and let $\{\mu^\varepsilon\}_{\varepsilon > 0}$ be the family of measures defined in Section 3. Then there exist a Mather measure μ and a nonnegative, symmetric $n \times n$ matrix $(m_{kj})_{k,j=1,\dots n}$ of Borel measures such that*

$$
\mu^\varepsilon \rightharpoonup \mu \quad \text{in the sense of measures up to subsequences,} \tag{5.1}
$$

$$
\operatorname{supp} \mu \text{ is compact,} \tag{5.2}
$$

and

$$
\int_{\mathbb{T}^n \times \mathbb{R}^n} \{\phi, H\}\, d\mu + \int_{\mathbb{T}^n \times \mathbb{R}^n} \phi_{p_k p_j}\, dm_{kj} = 0, \qquad \forall\, \phi \in C^2_c(\mathbb{T}^n \times \mathbb{R}^n). \tag{5.3}
$$

We call the matrix m_{kj} the dissipation measure.

Proof. First of all, we notice that since we have a uniform (in ε) Lipschitz estimate for the functions u^ε, there exists a compact set $K \subset \mathbb{T}^n \times \mathbb{R}^n$ such that

$$
\operatorname{supp} \mu^\varepsilon \subset K, \qquad \forall\, \varepsilon > 0.
$$

Moreover, up to subsequences, we have (5.1), that is

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{T}^n \times \mathbb{R}^n} \phi\, d\mu^\varepsilon \to \int_{\mathbb{T}^n \times \mathbb{R}^n} \phi\, d\mu,
$$

for every function $\phi \in C_c(\mathbb{T}^n \times \mathbb{R}^n)$, for some probability measure $\mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$. From what we said, it follows that

$$\operatorname{supp} \mu \subset K,$$

so that (5.1) and (5.2) are proved. To show (5.3), we need to pass to the limit in relation (3.6). First, let us focus on the second term of the aforementioned formula:

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \left[\frac{\varepsilon^2}{2} \phi_{x_i x_i} + \varepsilon^2 u^\varepsilon_{x_i x_j} \phi_{x_i p_j} + \frac{\varepsilon^2}{2} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_j} \phi_{p_k p_j} \right] d\mu^\varepsilon. \tag{5.4}$$

By the bounds of the previous section,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{T}^n \times \mathbb{R}^n} \left[\frac{\varepsilon^2}{2} \phi_{x_i x_i} + \varepsilon^2 u^\varepsilon_{x_i x_j} \phi_{x_i p_j} \right] d\mu^\varepsilon = 0.$$

However, as in [Eva], the last term in (5.4) does not vanish in the limit. In fact, through a subsequence, for every $k, j = 1, \dots, n$ we have

$$\frac{\varepsilon^2}{2} \int_{\mathbb{T}^n \times \mathbb{R}^n} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_j} \psi(x,p)\, d\mu^\varepsilon(x,p) \longrightarrow \int_{\mathbb{T}^n \times \mathbb{R}^n} \psi(x,p)\, dm_{kj}(x,p) \quad \forall\, \psi \in C(\mathbb{T}^n \times \mathbb{R}^n),$$

for some nonnegative, symmetric $n \times n$ matrix $(m_{kj})_{k,j=1,\dots n}$ of Borel measures. Passing to the limit as $\varepsilon \to 0$ in (3.6) condition (5.3) follows.

Let us show that μ satisfies conditions (a)–(c). As in [Eva] and [Tra], consider

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \Big(H(x,p) - \overline{H}^\varepsilon(P) \Big)^2 \, d\mu^\varepsilon(x,p) = \frac{\varepsilon^4}{4} \int_{\mathbb{T}^n \times \mathbb{R}^n} |\Delta u^\varepsilon(x)|^2 \, d\mu^\varepsilon(x,p) \longrightarrow 0$$

as $\varepsilon \to 0$, where we used (2.1) and (4.1). Therefore, (a) follows. Let us consider relation (3.6), and let us choose as test function $\phi = \varphi(u^\varepsilon)$. We get

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \varphi'(u^\varepsilon) D_x u^\varepsilon \cdot D_p H \, d\mu^\varepsilon + \varepsilon^2 \int_{\mathbb{T}^n \times \mathbb{R}^n} \left(\varphi'(u^\varepsilon) u^\varepsilon_{x_i x_i} + \varphi''(u^\varepsilon)(u^\varepsilon_{x_i})^2 \right) d\mu^\varepsilon = 0.$$

Passing to the limit as $\varepsilon \to 0$, we have

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \varphi'(u)(p - P) \cdot D_p H \, d\mu = 0.$$

Choosing $\varphi(u) = u$ we get (b). Finally, relation (c) follows by simply choosing in (5.3) test functions ϕ that do not depend on the variable p.

□

6. Support of the dissipation measures

6.1. **First identity involving H.** First recall that for any function $f:\mathbb{R}\to\mathbb{R}$ of class C^1

$$\{H, f(H)\} = 0,$$

and, furthermore, for any $\psi \in C^1(\mathbb{T}^n\times\mathbb{R}^n)$

$$\{H, \psi f(H)\} = \{H,\psi\}\, f(H).$$

Let $\lambda\in\mathbb{R}$. By choosing in (5.3) $\phi = \psi f(H)$, with $f(z) = e^{\lambda z}$ and $\psi \equiv 1$, we conclude that

$$0 = \int_{\mathbb{T}^n\times\mathbb{R}^n} e^{\lambda H}\left(\lambda H_{p_k}H_{p_j} + H_{p_k p_j}\right)\, dm_{kj}. \tag{6.1}$$

6.2. **On the support of μ.** For our convenience, we will write supp μ as

$$\operatorname{supp}\mu = \bigcup_{x\in\mathbb{T}^n} G(x),$$

where for every $\overline{x}\in\mathbb{T}^n$ we set $G(\overline{x}) := \operatorname{supp}\mu \cap \{(x,p)\in\mathbb{T}^n\times\mathbb{R}^n : x = \overline{x}\}$. We want to prove that

$$\operatorname{supp} m \subset \overline{\bigcup_{x\in\mathbb{T}^n} \operatorname{co} G(x)} =: K.$$

Here for every fixed $x\in\mathbb{T}^n$, with $\operatorname{co} G(x)$ we denote the convex hull in $\mathbb{R}^n$ (i.e. in the variable p) of the set $G(x)$. We remark that the closure in the right-hand side is taken in *all* $\mathbb{T}^n\times\mathbb{R}^n$.

For $\tau > 0$ sufficiently small, we can choose K_τ to be the open set in $\mathbb{T}^n\times\mathbb{R}^n$ such that $K\subset K_\tau$, $\operatorname{dist}(\partial K_\tau, K) < \tau$, and $K_\tau(x) := \{p\in\mathbb{R}^n :\ (x,p)\in K_\tau\}$ is convex for all $x\in\mathbb{T}^n$.
Given $x\in\mathbb{T}^n$, we can find a function $\eta_\tau(x,.) : \mathbb{R}^n\to\mathbb{R}$ and a *strictly convex* set $K_{2\tau}(x)\supset K_\tau(x)$ such that $\operatorname{dist}(\partial K_{2\tau}(x), K_\tau(x)) < \tau$ such that

- $\eta_\tau(x,p) = 0$ for $p\in K_\tau(x)$.
- $p\mapsto \eta_\tau(x,p)$ is convex.
- $p\mapsto \eta_\tau(x,p)$ is *uniformly convex* on $\mathbb{R}^n\setminus K_{2\tau}(x)$.

Summing up, $\eta_\tau(x,p) = 0$ on $K_\tau\supset K\supset \operatorname{supp}\mu$. Therefore

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \{\eta_\tau, H\} d\mu = 0.$$

Combining with (5.3),

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} (\eta_\tau)_{p_k p_j} dm_{kj} = 0,$$

which implies $\operatorname{supp} m\subset \bigcup_{x\in\mathbb{T}^n} K_{2\tau}(x)$. Letting $\tau\to 0$, we finally get the desired result, which is

$$\operatorname{supp} m \subset \overline{\bigcup_{x\in\mathbb{T}^n} \operatorname{co} G(x)} =: K.$$

Corollary 6.1.

$$\operatorname{supp} m \subset \overline{\operatorname{co}\{H(x,p) \le \overline{H}\}}.$$

7. Averaging

In this section we prove some additional estimates concerning the averaging.

7.1. **Rotation Number.** Let us start with a definition.

Definition 7.1. We define the *rotation number* ρ_0 as

$$\rho_0 := \lim_{\varepsilon\to 0}\lim_{T\to+\infty} E\left[\frac{\mathbf{x}^\varepsilon(T)-\mathbf{x}^\varepsilon(0)}{T}\right].$$

The following theorem gives a formula for the rotation number.

Theorem 7.2. *There holds*

$$\rho_0 = \int_{\mathbb{T}^n\times\mathbb{R}^n} D_pH\, d\mu. \tag{7.1}$$

Moreover, defining for every $\varepsilon > 0$ the variable $\mathbf{X}^\varepsilon := \mathbf{x}^\varepsilon + D_P u^\varepsilon(\mathbf{x}^\varepsilon)$, we have

$$E\left[\frac{\mathbf{X}^\varepsilon(T)-\mathbf{X}^\varepsilon(0)}{T}\right] = -D_P\overline{H}^\varepsilon(P), \tag{7.2}$$

and

$$\lim_{T\to+\infty} E\left[\frac{\left(\mathbf{X}^\varepsilon(T)-\mathbf{X}^\varepsilon(0)+D_P\overline{H}^\varepsilon(P)T\right)^2}{T}\right] \le 2\,n\,\varepsilon^2 + 2\int_{\mathbb{T}^n}|D_Pu^\varepsilon|^2\,d\theta_{\mu^\varepsilon}$$
$$+\,2\int_{\mathbb{T}^n}|D_pH - D_P\overline{H}^\varepsilon|^2\,d\theta_{\mu^\varepsilon}.$$

Proof. Choosing $\phi(x) = x_i$ with $i = 1,2,3$ in (3.5) we obtain

$$E\left[\frac{\mathbf{x}^\varepsilon(T)-\mathbf{x}^\varepsilon(0)}{T}\right] = -E\left[\frac{1}{T}\int_0^T D_pH\,dt\right].$$

Passing to the limit as $T\to+\infty$

$$\rho_\varepsilon := \lim_{T\to+\infty} E\left[\frac{\mathbf{x}^\varepsilon(T)-\mathbf{x}^\varepsilon(0)}{T}\right] = \int_{\mathbb{T}^n\times\mathbb{R}^n} D_pH\,d\mu^\varepsilon.$$

We get (7.1) by letting ε go to zero.

To prove (7.2), recalling Itô's formula (3.2) we compute

$$\begin{aligned} d\mathbf{X}^\varepsilon &= d\mathbf{x}^\varepsilon + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)\,d\mathbf{x}^\varepsilon + \frac{\varepsilon^2}{2}D_P\Delta u^\varepsilon(\mathbf{x}^\varepsilon)\,dt \\ &= \left(-D_pH(\mathbf{x}^\varepsilon,\mathbf{p}^\varepsilon)(I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)) + \frac{\varepsilon^2}{2}D_P\Delta u^\varepsilon(\mathbf{x}^\varepsilon)\right)dt + \varepsilon(I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon))\,dw_t, \end{aligned}$$

where in the last equality we used (3.1). By differentiating equation (2.1) w.r.t. P we obtain

$$-D_pH(\mathbf{x}^\varepsilon,\mathbf{p}^\varepsilon)(I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)) + \frac{\varepsilon^2}{2}D_P\Delta u^\varepsilon(\mathbf{x}^\varepsilon) = -D_P\overline{H}^\varepsilon(P), \tag{7.3}$$

so that

$$d\mathbf{X}^\varepsilon = -D_P\overline{H}^\varepsilon(P)\,dt + \varepsilon(I + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon))\,dw_t. \tag{7.4}$$

Using the fact that

$$E\left[\int_0^T \varepsilon(I + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon))\,dw_t\right] = 0,$$

(7.2) follows.

Finally, using once again Itô's formula (3.2) and relation (7.4) we can write

$$\begin{aligned}
&d\left[\left(\mathbf{X}^\varepsilon(t) - \mathbf{X}^\varepsilon(0) + D_P\overline{H}^\varepsilon(P)t\right)^2\right] \\
&= 2\left(\mathbf{X}^\varepsilon(t) - \mathbf{X}^\varepsilon(0) + D_P\overline{H}^\varepsilon(P)t\right)(d\,\mathbf{X}^\varepsilon + D_P\overline{H}^\varepsilon(P)\,dt) + \varepsilon^2|I + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2\,dt \\
&= 2\,\varepsilon\left(\mathbf{X}^\varepsilon(t) - \mathbf{X}^\varepsilon(0) + D_P\overline{H}^\varepsilon(P)t\right)(I + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon))\,dw_t + \varepsilon^2|I + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2\,dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&E\left[\left(\mathbf{X}^\varepsilon(T) - \mathbf{X}^\varepsilon(0) + D_P\overline{H}^\varepsilon(P)T\right)^2\right] \\
&= E\left[\int_0^T 2\,\varepsilon\left(\mathbf{X}^\varepsilon(t) - \mathbf{X}^\varepsilon(0) + D_P\overline{H}^\varepsilon(P)t\right)(I + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon))\,dw_t + \int_0^T \varepsilon^2|I + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2\,dt\right] \\
&= E\left[\int_0^T \varepsilon^2|I + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2\,dt\right].
\end{aligned}$$

Dividing by T and letting T go to infinity

$$\begin{aligned}
&\lim_{T\to+\infty} E\left[\frac{\left(\mathbf{X}^\varepsilon(T) - \mathbf{X}^\varepsilon(0) + D_P\overline{H}^\varepsilon(P)T\right)^2}{T}\right] = \lim_{T\to+\infty} E\left[\int_0^T \frac{\varepsilon^2|I + D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2}{T}\,dt\right] \\
&= \varepsilon^2\int_{\mathbb{T}^n}|I + D^2_{Px}u^\varepsilon|^2\,d\theta_{\mu^\varepsilon} \le 2\,n\,\varepsilon^2 + 2\,\varepsilon^2\int_{\mathbb{T}^n}|D^2_{Px}u^\varepsilon|^2\,d\theta_{\mu^\varepsilon} \\
&\le 2\,n\,\varepsilon^2 + 2\int_{\mathbb{T}^n}|D_Pu^\varepsilon|^2\,d\theta_{\mu^\varepsilon} + 2\int_{\mathbb{T}^n}|D_pH - D_P\overline{H}^\varepsilon|^2\,d\theta_{\mu^\varepsilon},
\end{aligned}$$

where we used (4.2).

□

7.2. **Averaging.** In this subsection we show in a formal way how relation (1.3) is "far" from being a real change of variables.

Let us set $w^\varepsilon(x,P) := P\cdot x + u^\varepsilon(x,P)$, where $u^\varepsilon(x,P)$ is a $\mathbb{Z}^n$-periodic viscosity solution of (1.17), and let $k \in \mathbb{Z}^n$. Recalling identity (3.7) with

$$\varphi(x) = e^{2\pi i k\cdot D_P w^\varepsilon(x,P)}$$

we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{T}^n} L^{\varepsilon,P} e^{2\pi i k\cdot D_P w^\varepsilon}\, d\theta_{\mu^\varepsilon} \\ &= 2\pi i \int_{\mathbb{T}^n} e^{2\pi i k\cdot D_P w^\varepsilon} \left[L^{\varepsilon,P}\left(k\cdot D_P w^\varepsilon\right) - \pi i \varepsilon^2 |D_x(k\cdot D_P w^\varepsilon)|^2\right] d\theta_{\mu^\varepsilon} \\ &= 2\pi i \int_{\mathbb{T}^n} e^{2\pi i k\cdot D_P w^\varepsilon} \left[k\cdot D_P \overline{H}^\varepsilon - \pi i \varepsilon^2 |D_x(k\cdot D_P w^\varepsilon)|^2\right] d\theta_{\mu^\varepsilon}, \end{aligned}$$

where we used (4.8) and the fact that $w^\varepsilon = P\cdot x + u^\varepsilon$. Thus, thanks to estimate (4.2)

$$\begin{aligned} &\left| (k\cdot D_P \overline{H}^\varepsilon) \int_{\mathbb{T}^n} e^{2\pi i k\cdot D_P w^\varepsilon}\, d\theta_{\mu^\varepsilon} \right| \le \pi\varepsilon^2 \int_{\mathbb{T}^n} |D_x(k\cdot D_P w^\varepsilon)|^2\, d\theta_{\mu^\varepsilon} \\ &\le 2\pi |k|^2 \left(\varepsilon^2 + \varepsilon^2 \int_{\mathbb{T}^n} |D^2_{Px} u^\varepsilon|^2\, d\theta_{\mu^\varepsilon} \right) \\ &\le 2\pi |k|^2 \left(\varepsilon^2 + \int_{\mathbb{T}^n} |D_P u^\varepsilon|^2\, d\theta_{\mu^\varepsilon} + \int_{\mathbb{T}^n} |D_p H - D_P \overline{H}^\varepsilon|^2\, d\theta_{\mu^\varepsilon} \right). \end{aligned}$$

8. Compensated compactness

In this section, some analogs of Compensated compactness and Div-Curl lemma introduced by Murat and Tartar in the context of Conservation laws (see [Eva90], [Tar79]) will be studied to understand more about the support of the Mather measure μ. Similar analogs are also studied by Evans to understand more about the shock nature of non-convex Hamilton-Jacobi equations in [Eva].

Obviously, what we are doing here is quite different from the original Murat and Tartar work (see [Tar79]), since we work on the support of the measure θ_{μ^ϵ}. Besides, the methods work on arbitrary dimensional space $\mathbb{R}^n$ while usual Compensated compactness and Div-Curl lemma in the context of Conservation laws can only deal with the case $n = 1, 2$. However, we can only derive one single relation and this is not enough to characterize the support of μ as in the convex case.

Let ϕ be a smooth function from $\mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$, and let $\rho^\epsilon = \{\phi, H\}\theta_{\mu^\epsilon} + \frac{\epsilon^2}{2}\phi_{p_j p_k} u^\epsilon_{x_i x_j} u^\epsilon_{x_i x_k} \theta_{\mu^\epsilon}$. By (3.6) and (4.1), there exists $C > 0$ such that

$$\int_{\mathbb{T}^n} |\rho^\epsilon| dx \le C.$$

So, up to passing to some subsequence, if necessary, we may assume that $\rho^\epsilon \rightharpoonup \rho$ as a (signed) measure.

By (5.3), $\rho(\mathbb{T}^n) = 0$. We have the following theorem

Theorem 8.1. *We have the following properties*

(i)

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} D_p H.(p-P)\phi(x,p) d\mu = \int_{\mathbb{T}^n} u d\rho. \tag{8.1}$$

(ii) *For* $\eta \in C^1(\mathbb{T}^n)$,

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} D_pH.D\eta\phi(x,p)d\mu = \int_{\mathbb{T}^n}\eta d\rho. \tag{8.2}$$

Proof

Let $w^\epsilon = \phi(x, P + Du^\epsilon)$. Notice first that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} D_pH.(p-P)\phi(x,p)d\mu = \lim_{\epsilon\to 0}\int_{\mathbb{T}^n} D_pH(x,P+Du^\epsilon).Du^\epsilon w^\epsilon d\theta_{\mu^\epsilon}.$$

Integrating by parts the right hand side of the above equality we obtain

$$\int_{\mathbb{T}^n} D_pH(x,P+Du^\epsilon).Du^\epsilon w^\epsilon d\theta_{\mu^\epsilon} = -\int_{\mathbb{T}^n} u^\epsilon \mathrm{div}(D_pHw^\epsilon\theta_{\mu^\epsilon})dx$$
$$= -\int_{\mathbb{T}^n} u^\epsilon(\mathrm{div}(D_pH\theta_{\mu^\epsilon})w^\epsilon + D_pH.Dw^\epsilon\theta_{\mu^\epsilon})dx = \int_{\mathbb{T}^n} u^\epsilon(\frac{\epsilon^2}{2}\Delta\theta_{\mu^\epsilon}w^\epsilon - D_pH.Dw^\epsilon\theta_{\mu^\epsilon})dx.$$

By several computations by using (2.1), we get

$$D_pH.Dw^\epsilon = -\{\phi, H\} + \frac{\epsilon^2}{2}\phi_{p_i}\Delta u^\epsilon_{x_i}.$$

Hence

$$\frac{\epsilon^2}{2}\Delta\theta_{\mu^\epsilon}w^\epsilon - D_pH.Dw^\epsilon\theta_{\mu^\epsilon} = \frac{\epsilon^2}{2}\Delta\theta_{\mu^\epsilon}w^\epsilon + \{\phi,H\}\theta_{\mu^\epsilon} - \frac{\epsilon^2}{2}\phi_{p_i}\Delta u^\epsilon_{x_i}\theta_{\mu^\epsilon}$$
$$=\frac{\epsilon^2}{2}\Delta w^\epsilon\theta_{\mu^\epsilon} + \frac{\epsilon^2}{2}(\mathrm{div}(D\theta_{\mu^\epsilon}w^\epsilon) - \mathrm{div}(Dw^\epsilon\theta_{\mu^\epsilon})) + \{\phi,H\}\theta_{\mu^\epsilon} - \frac{\epsilon^2}{2}\phi_{p_i}\Delta u^\epsilon_{x_i}\theta_{\mu^\epsilon}$$
$$=\frac{\epsilon^2}{2}(\phi_{p_jp_k}u^\epsilon_{x_ix_j}u^\epsilon_{x_ix_k} + \phi_{p_jx_i}u^\varepsilon_{x_jx_i} + \phi_{x_ix_i} + \phi_{p_i}\Delta u^\epsilon_{x_i})\theta_{\mu^\epsilon}$$
$$+\frac{\epsilon^2}{2}(\mathrm{div}(D\theta_{\mu^\epsilon}w^\epsilon) - \mathrm{div}(Dw^\epsilon\theta_{\mu^\epsilon})) + \{\phi,H\}\theta_{\mu^\epsilon} - \frac{\epsilon^2}{2}\phi_{p_i}\Delta u^\epsilon_{x_i}\theta_{\mu^\epsilon}$$
$$=\rho^\epsilon + \frac{\epsilon^2}{2}\phi_{x_ix_i}\theta_{\mu^\epsilon} + \frac{\varepsilon^2}{2}\phi_{p_jx_i}u^\varepsilon_{x_jx_i} + \frac{\epsilon^2}{2}(\mathrm{div}(D\theta_{\mu^\epsilon}w^\epsilon) - \mathrm{div}(Dw^\epsilon\theta_{\mu^\epsilon})).$$

Therefore

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} D_pH.(p-P)\phi(x,p)d\mu$$
$$= \lim_{\epsilon\to 0}\int_{\mathbb{T}^n} u^\epsilon[\rho^\epsilon + \frac{\epsilon^2}{2}\phi_{x_ix_i}\theta_{\mu^\epsilon} + \frac{\varepsilon^2}{2}\phi_{p_jx_i}u^\varepsilon_{x_jx_i} + \frac{\epsilon^2}{2}(\mathrm{div}(D\theta_{\mu^\epsilon}w^\epsilon) - \mathrm{div}(Dw^\epsilon\theta_{\mu^\epsilon}))]dx. \tag{8.3}$$

Since u^ϵ converges uniformly to u,

$$\lim_{\epsilon\to 0}\int_{\mathbb{T}^n} u^\epsilon\rho^\epsilon dx = \int_{\mathbb{T}^n} ud\rho.$$

The second term in the right hand side of (8.3) obviously converges to 0 as $\epsilon \to 0$. The third term also tends to 0 by (4.1).

Let's look at the last term

$$
\begin{aligned}
&|\lim_{\epsilon\to 0}\frac{\epsilon^2}{2}\int_{\mathbb{T}^n} u^\epsilon(\mathrm{div}(D\theta_{\mu^\epsilon}w^\epsilon)-\mathrm{div}(Dw^\epsilon\theta_{\mu^\epsilon}))dx| = |\lim_{\epsilon\to 0}\frac{\epsilon^2}{2}\int_{\mathbb{T}^n} -Du^\epsilon.D\theta_{\mu^\epsilon}w^\epsilon + Du^\epsilon.Dw^\epsilon\theta_{\mu^\epsilon}dx|\\
=&|\lim_{\epsilon\to 0}\frac{\epsilon^2}{2}\int_{\mathbb{T}^n}\mathrm{div}(Du^\epsilon w^\epsilon)\theta_{\mu^\epsilon} + Du^\epsilon.Dw^\epsilon\theta_{\mu^\epsilon}dx| = |\lim_{\epsilon\to 0}\frac{\epsilon^2}{2}\int_{\mathbb{T}^n}(\Delta u^\epsilon w^\epsilon + 2Du^\epsilon.Dw^\epsilon)\theta_{\mu^\epsilon}dx|\\
\le&\lim_{\epsilon\to 0}C\epsilon^2\int_{\mathbb{T}^n}|D^2u^\epsilon|\theta_{\mu^\epsilon}dx \le \lim_{\epsilon\to 0}C\epsilon = 0.
\end{aligned}
$$

We get (8.1). (8.2) can be derived similarly.

Remark 8.2. If we have in addition that u is C^1 on the support of μ then by (8.1) and (8.2)

$$\int_{\mathbb{T}^n} D_pH.(p-P-Du)\phi d\mu = 0,$$

for all ϕ. Therefore, $D_pH.(p-P-Du)=0$ on the support of μ. This is quite a good relation and a partial result of the property (2) of the Mather measure. Obviously we cannot have $p=P+Du$ on the support of the Mather measure like in the case where H is uniformly convex.

9. Examples

In this section, we study non-trivial examples where the Mather measure μ is invariant under the Hamiltonian dynamics. Notice that the Mather measure μ is invariant under the Hamiltonian dynamics if and only if the *dissipation measures* (m_{kj}) vanish by (5.3). We do strongly believe that the *dissipation measures* may not vanish in general and they record the jump (or the difference) of the gradient Du along the shocks like the case appeared in [Eva]. The interesting question is that under which suitable conditions can we have such property? Here we provide some partial answers to this question by studying some non-trivial examples.

9.1. H **is uniformly convex.** There exists $\alpha>0$ so that $D^2_{pp}H\ge\alpha>0$.
Let $\lambda=0$ in (6.1) then

$$0=\int_{\mathbb{T}^n\times\mathbb{R}^n} H_{p_kp_j}dm_{kj},$$

which implies $m_{kj}=0$ for all $1\le k,j\le n$. We then can do the same steps as in [EG01] to get that μ also satisfies (2).

9.2. **Uniformly convex conservation law.** Suppose that there exists $F(p,x)$, strictly convex in p such that $\{F,H\}$=0. Then $m=0$.

9.3. **Some special non-convex cases.** The cases we consider here are somehow the variants of the uniformly convex case.

In (5.3), if there exist ϕ uniformly convex and f, g smooth real functions such that $\phi = f(H)$ or $H = g(\phi)$ then $m_{kj} = 0$ for all k, j.

In particular, if $H = g(\phi)$ where g is increasing then H is level set quasi-convex.

One explicit example of the above variants would be $H(x,p) = (|p|^2 + V(x))^2$ where $V : \mathbb{T}^n \to \mathbb{R}$ smooth and may have negative values. Then $H(x,p)$ is not convex in p anymore. We can choose $\phi(x,p) = |p|^2 + V(x)$ then $H(x,p) = (\phi(x,p))^2$ and ϕ is uniformly convex in p. Therefore, we get μ is invariant under the Hamiltonian dynamics.

9.4. **The case when** $n = 1$**.** Let's consider the case $H(x,p) = H(p) + V(x)$.

Assume that there exists $p_0 \in \mathbb{R}$ such that $H'(p) = 0$ if and only if $p = p_0$ and $H''(p_0) \neq 0$. Notice that this condition is quite weak and $H(p)$ does not need to be convex. Obviously, uniform convexity of H implies this condition.

We will show that $m_{11} = 0$, which implies μ is invariant under the Hamiltonian dynamics.

Since H is superlinear, therefore $H'(p) > 0$ for $p > p_0$, $H'(p) < 0$ for $p < p_0$ and hence $H''(p_0) > 0$. There exists a neighborhood $(p_0 - r, p_0 + r)$ of p_0 such that

$$H''(p) > \frac{H''(p_0)}{2}, \qquad \forall\, p \in (p_0 - r, p_0 + r).$$

And since the support of m_{11} is bounded, we may assume

$$\operatorname{supp}(m_{11}) \subset \mathbb{T} \times [-M, M],$$

for some $M > 0$ large enough. We can choose M large so that $(p_0 - r, p_0 + r) \subset (-M, M)$.

Since $|H'(p)|^2 > 0$ for $p \in [-M, M] \setminus (p_0 - r, p_0 + r)$ and $[-M, M] \setminus (p_0 - r, p_0 + r)$ is compact, there exists $\gamma > 0$ such that

$$|H'(p)|^2 \geq \gamma > 0, \qquad \forall\, p \in [-M, M] \setminus (p_0 - r, p_0 + r).$$

Hence, by choosing $\lambda \gg 0$

$$\lambda |H'(p)|^2 + H''(p) \geq \frac{H''(p_0)}{2}, \qquad \forall\, p \in [-M, M],$$

which shows $m_{11} = 0$ by (6.1).

9.5. **The case we have more conserved quantities.** Let's consider

$$H(x,p) = H(p) + V(x_1 + ... + x_n),$$

where $V : \mathbb{T} \to \mathbb{R}$ is smooth.

For $k \neq j$, define $\Phi^{kj} = p_k - p_j$. It is easy to see that $\{H, \Phi^{kj}\} = 0$ for any $k \neq j$.

Therefore $\{H, (\Phi^{kj})^2\} = 0$ for any $k \neq j$.

For fixed $k \neq j$, let $\phi = (\Phi^{kj})^2$ in (5.3) then

$$2\int_{\mathbb{T}^n \times \mathbb{R}^n} (m_{kk} - 2m_{kj} + m_{jj}) dxdp = 0.$$

Since the matrix of *dissipation measures* (m_{kj}) is non-negative definite, we therefore also have $m_{kk} - 2m_{kj} + m_{jj} \geq 0$. Thus, $m_{kk} - 2m_{kj} + m_{jj} = 0$ for any $k \neq j$.

Now take $\xi = (\xi_1, ..., \xi_n)$ where $\xi_k = 1 + \epsilon, \xi_j = -1$ and $\xi_i = 0$ otherwise. We have

$$0 \leq m_{kj}\xi_k\xi_j = (1+\epsilon)^2 m_{kk} - 2(1+\epsilon)m_{kj} + m_{jj} = 2\epsilon(m_{kk} - m_{kj}) + \epsilon^2 m_{kk}.$$

Dividing both sides of the inequality above by ϵ and let $\epsilon \to 0$,

$$m_{kk} - m_{kj} \geq 0.$$

Similarly, $m_{jj} - m_{kj} \geq 0$. Thus, $m_{kk} - m_{kj} = m_{jj} - m_{kj} = 0$ for all $k \neq j$.

Hence, there exists a non-negative measure m such that

$$m_{kj} = m \geq 0, \qquad \forall\, k, j.$$

Therefore, (6.1) becomes

$$0 = \int_{\mathbb{T}^n \times \mathbb{R}^n} e^{\lambda H} (\lambda (\sum_j H_{p_j})^2 + \sum_{j,k} H_{p_j p_k}) dm$$

We here point out two cases which guarantee that $m = 0$. In the first case assuming additionally that $H(p) = H_1(p_1) + ... H_n(p_n)$ and $H_2, ..., H_n$ are convex but not necessarily uniformly convex (their graphs may have flat regions) and H_1 is uniformly convex then we still have $m = 0$.

In the second case, suppose that $H(p) = H(|p|)$ where $H : [0, \infty) \to \mathbb{R}$ is smooth, $H'(0) = 0, H''(0) > 0$ and $H'(s) > 0$ for $s > 0$. Notice that H is not necessarily convex. This example is quite similar to the example above when $n = 1$.

For $p \neq 0$

$$\lambda(\sum_j H_{p_j})^2 + \sum_{j,k} H_{p_j p_k} = n\frac{H'}{|p|} + \frac{(p_1 + ... + p_n)^2}{|p|^2}(\lambda (H')^2 + H'' - \frac{H'}{|p|}),$$

and at $p = 0$

$$\lambda(\sum_j H_{p_j}(0))^2 + \sum_{j,k} H_{p_j p_k}(0) = nH''(0) > 0.$$

So, we can choose some $r > 0$ small enough so that for $|p| < r$

$$\lambda(\sum_j H_{p_j})^2 + \sum_{j,k} H_{p_j p_k} > \frac{n}{2} H''(0) > 0.$$

Since the support of m is bounded, there exists $M > 0$ large enough

$$\text{supp}(m) \subset \mathbb{T}^n \times \{p : |p| \le M\}.$$

Since $\min_{s\in[r,M]} H'(s) > 0$, by choosing $\lambda \gg 0$, we finally have for $|p| \le M$

$$\lambda(\sum_j H_{p_j})^2 + \sum_{j,k} H_{p_j p_k} \ge \beta > 0,$$

for $\beta = \frac{n}{2} \min\{H''(0), \frac{\min_{s\in[r,M]} H'(s)}{M}\}$.
Thus $m = 0$, and therefore μ is invariant under the Hamiltonian dynamics.

9.6. **Quasi-convex Hamiltonians: a special case.** Let's consider

$$H(x,p) = H(|p|) + V(x),$$

where $H : [0,\infty) \to \mathbb{R}$ is smooth, $H'(0) = 0, H''(0) > 0$ and $H'(s) > 0$ for $s > 0$.
Once again, notice that H is not necessarily convex. We here will show that $(m_{jk}) = 0$. For $p \neq 0$ then

$$(\lambda H_{p_j} H_{p_k} + H_{p_j p_k}) m_{jk} = \frac{H'}{|p|}(m_{11} + ... + m_{nn}) + (\lambda (H')^2 + H'' - \frac{H'}{|p|})\frac{p_j p_k m_{jk}}{|p|^2}.$$

For the symmetric, non-negative definite matrix $m = (m_{jk})$ we have the following inequality

$$0 \le p_j p_k m_{jk} \le |p|^2 \text{trace}(m) = |p|^2 (m_{11} + ... + m_{nn}).$$

There exists $r > 0$ small enough so that for $|p| < r$

$$\frac{H'}{|p|} > \frac{3}{4} H''(0); |\frac{H'}{|p|} - H''| < \frac{1}{4} H''(0).$$

Hence for $|p| < r$

$$(\lambda H_{p_j} H_{p_k} + H_{p_j p_k}) m_{jk} \ge \frac{1}{2} H''(0)(m_{11} + ... + m_{nn}).$$

Since the support of (m_{jk}) are bounded, there exists $M > 0$ large enough

$$\text{supp}(m_{jk}) \subset \mathbb{T}^n \times \{p : |p| \le M\}, \qquad \forall\ j,k.$$

Since $\min_{s\in[r,M]} H'(s) > 0$, by choosing $\lambda \gg 0$ we finally have for $|p| \le M$

$$(\lambda H_{p_j} H_{p_k} + H_{p_j p_k}) m_{jk} \ge \beta (m_{11} + ... + m_{nn}),$$

for $\beta = \min\{\frac{H''(0)}{2}, \frac{\min_{s\in[r,M]} H'(s)}{M}\} > 0$.
We then must have $m_{11} + ... m_{nn} = 0$, which implies $(m_{jk}) = 0$.

Thus, μ is invariant under the Hamiltonian dynamics in this case.

We now derive the property (2) of μ rigorously.

Since the support of μ is also bounded, we can use the similar steps as above to show that $\phi(x,p) = e^{\lambda H(x,p)}$ is uniformly convex in $\mathbb{T}^n \times \bar{B}(0,M) \supset \operatorname{supp}(\mu)$ for some λ large enough.

More precisely,

$$\phi_{p_j p_k}\xi_j\xi_k \geq e^{\lambda H}\beta|\xi|^2, \quad \xi \in \mathbb{R}^n,\ (x,p) \in \mathbb{T}^n \times \bar{B}(0,M),$$

for β chosen as above. Then doing the same steps as in [EG01], we get μ satisfies (2).

There is another simple approach to prove (2) by using the properties we get in this non-convex setting. Let's just assume that u is C^1 on the support of μ.

By Remark 8.2, it follows that $D_pH.(p - P - Du) = 0$ on support of μ. And since $D_pH(x,p) = H'(|p|)\dfrac{p}{|p|}$ for $p \neq 0$ and $H'(|p|) > 0$, we then have $p.(p - P - Du) = 0$ on support of μ. Hence $|p|^2 = p.(P + Du)$ on $\operatorname{supp}(\mu)$.

Besides, $H(x,p) = H(x, P + Du(x)) = \overline{H}(P)$ on $\operatorname{supp}(\mu)$ by property (a) of Mather measure and the assumption that u is C^1 on $\operatorname{supp}(\mu)$. It follows that $H(|p|) = H(|P + Du|)$. Therefore, $|p| = |P + Du|$ by the fact that $H(s)$ is strictly increasing.

So we have $|p|^2 = p.(P + Du)$ and $|p| = |P + Du|$ on $\operatorname{supp}(\mu)$, which implies $p = P + Du$ on $\operatorname{supp}(\mu)$, which is the property (2) of μ.

9.7. **Quasi-convex Hamiltonians.** We treat now the general case of uniformly quasi-convex Hamiltonians. We start with a definition.

Definition 9.1. A smooth set $A \subset \mathbb{R}^n$ is said to be *strongly convex with convexity constant* c if there exists a positive constant c with the following property. For every $p \in \partial A$ there exist an orthogonal coordinate system $(q_1, \ldots, q_n)$ centered at p, and a coordinate rectangle $R = (a_1, b_1) \times \ldots \times (a_n, b_n)$ containing p such that $T_p\partial A = \{q_n = 0\}$ and $A \cap R \subset \{q \in R : c\sum_{i=1}^{n-1}|q_i|^2 \leq q_n \leq b_n\}$.

The previous definition can be stated in the following equivalent way, by requiring that for every $p \in \partial A$

$$(\mathbf{B}_p\mathbf{v}) \cdot \mathbf{v} \geq c|\mathbf{v}|^2 \quad \text{for every } \mathbf{v} \in T_p\partial A,$$

where $\mathbf{B}_p : T_p\partial A \times T_p\partial A \to \mathbb{R}$ is the second fundamental form of ∂A at p.

We consider in this subsection strongly quasi-convex Hamiltonians. That is, we assume that there exists $c > 0$ such that

(j) $\{p \in \mathbb{T}^n : H(x,p) \leq a\}$ is strongly convex with convexity constant c for every $a \in \mathbb{R}$ and for every $x \in \mathbb{T}^n$.

In addition, we suppose that there exists $\alpha \in \mathbb{R}$ such that for every $x \in \mathbb{T}^n$

(jj) There exists unique $\overline{p} \in \mathbb{R}^n$ s.t. $D_pH(x,\overline{p}) = 0$, and

$$D^2_{pp}H(x,\overline{p}) \geq \alpha.$$

Notice that the special case just presented fits into this definition, with level sets given by spheres. We will show that under hypotheses (j)–(jj) there exists $\lambda > 0$ such that

$$\lambda\, D_pH \otimes D_pH + D^2_{pp}H \quad \text{ is positive definite.}$$

From this, thanks to relation(6.1), we conclude that $m_{kj} = 0$. First, we state a well-known result in literature. We give the proof below, for the convenience of the reader.

Proposition 9.2. *Let (j)–(jj) be satisfied, and let $(x^*,p^*) \in \mathbb{T}^n \times \mathbb{R}^n$ be such that $D_pH(x^*,p^*) \neq 0$. Then*

$$D_pH(x^*,p^*) \perp T_{p^*}\mathcal{C} \quad \text{ and } \quad D^2_{pp}H(x^*,p^*) = |D_pH(x^*,p^*)|\mathbf{B}_{p^*}, \tag{9.1}$$

where $\mathbf{B}_{p^*}$ denotes the second fundamental form of the level set

$$\mathcal{C} := \{p \in \mathbb{R}^n : H(x^*,p) = H(x^*,p^*)\}$$

at the point p^*.

Proof. By the smoothness of H, there exists a neighborhood $U \subset \mathbb{R}^n$ of p^* and n smooth functions $\nu : U \to \mathcal{S}^{n-1}$, $\tau_i : U \to \mathcal{S}^{n-1}$, $i = 1,\ldots,n-1$, such that for every $p \in U$ the vectors $\{\tau_1(p),\ldots,\tau_{n-1}(p),\nu(p)\}$ are a smooth orthonormal basis of $\mathbb{R}^n$, and for every $p \in U \cap \mathcal{C}$ $\tau_1(p),\ldots,\tau_{n-1}(p) \in T_p\mathcal{C}$. Let now $i,j \in \{1,\ldots,n-1\}$ be fixed. Since

$$H(x^*,p) = a \quad \forall\, p \in U,$$

differentiating w.r.t $\tau_i(p)$ we have

$$D_pH(x^*,p)\cdot\tau_i(p) = 0 \quad \forall\, p \in U \cap \mathcal{C}. \tag{9.2}$$

Computing last relation at $p = p^*$ we get that $D_pH(x^*,p^*) \perp T_{p^*}\mathcal{C}$. Differentiating (9.2) along the direction $\tau_j(p)$ and computing at $p = p^*$

$$\left(D^2_{pp}H(x^*,p^*)\tau_j(p^*)\right)\cdot\tau_i(p^*) + D_pH(x^*,p^*)\cdot\left(D_p\tau_i(p^*)\tau_j(p^*)\right) = 0. \tag{9.3}$$

Notice that by differentiating along the direction $\tau_j(p)$ the identity $\tau_i(p)\cdot\nu(p)=0$ and computing at p^* we get

$$(D_p\tau_i(p^*)\tau_j(p^*))\cdot\nu(p^*) = -\left(D_p\nu(p^*)\tau_j(p^*)\right)\cdot\tau_i(p^*).$$

Plugging last relation into (9.3), and choosing $\nu(p^*)$ oriented in the direction of $D_pH(x^*,p^*)$ we have

$$\begin{aligned}&\left(D^2_{pp}H(x^*,p^*)\tau_j(p^*)\right)\cdot\tau_i(p^*) = -|D_pH(x^*,p^*)|\left(D_p\tau_i(p^*)\tau_j(p^*)\right)\cdot\nu(p^*)\\ &=|D_pH(x^*,p^*)|\left(D_p\nu(p^*)\tau_j(p^*)\right)\cdot\tau_i(p^*) = |D_pH(x^*,p^*)|\left(\mathbf{B}_{p^*}\tau_j(p^*)\right)\cdot\tau_i(p^*).\end{aligned}$$

□

For every vector $v\in\mathbb{R}^n$, we consider the decomposition

$$v = v_\parallel\mathbf{v}^\parallel + v_\perp\mathbf{v}^\perp,$$

with $v^\parallel, v^\perp\in\mathbb{R}$, $|\mathbf{v}^\parallel| = |\mathbf{v}^\perp| = 1$, $\mathbf{v}^\parallel\in T_{p^*}\mathcal{C}$, and $\mathbf{v}^\perp\in(T_{p^*}\mathcal{C})^\perp$. By hypothesis (jj) and by the smoothness of H, there exist $\tau>0$ and $\alpha'\in(0,\alpha)$, independent of (x,p), such that

$$D^2_{pp}H(x,p)\ge\alpha' \quad \text{for every } (x,p)\in\{|D_pH|\le\tau\}.$$

Let us now consider two subcases:

Case 1: $(x,p)\in\{|D_pH|\le\tau\}$

First of all, notice that

$$\lambda\, D_pH\otimes D_pHv\cdot v = \lambda|D_pH\cdot v|^2 = \lambda\, v_\perp^2\,|D_pH|^2.$$

Then, we have

$$(\lambda\, D_pH\otimes D_pH + D^2_{pp}H)v\cdot v = \lambda\, v_\perp^2\,|D_pH|^2 + (D^2_{pp}Hv\cdot v)\ge\alpha'|v|^2.$$

Case 2: $(x,p)\in\{|D_pH|>\tau\}$

In this case

$$\begin{aligned}D^2_{pp}Hv\cdot v &= v_\parallel^2(D^2_{pp}H\mathbf{v}^\parallel\cdot\mathbf{v}^\parallel) + 2v_\parallel v_\perp(D^2_{pp}H\mathbf{v}^\parallel\cdot\mathbf{v}^\perp) + v_\perp^2(D^2_{pp}H\mathbf{v}^\perp\cdot\mathbf{v}^\perp)\\ &\ge c\, v_\parallel^2|D_pH| + 2v_\parallel v_\perp(D^2_{pp}H\mathbf{v}^\parallel\cdot\mathbf{v}^\perp) + v_\perp^2(D^2_{pp}H\mathbf{v}^\perp\cdot\mathbf{v}^\perp).\end{aligned}$$

By (5.2) we have

$$|D^2_{pp}H|\le C \quad \text{along } \operatorname{supp}\mu.$$

Thus,

$$\begin{aligned}
&(\lambda\, D_pH \otimes D_pH + D^2_{pp}H)v \cdot v \\
&\geq \lambda\, v_\perp^2\, |D_pH|^2 + c\, v_\parallel^2 |D_pH| + 2v_\parallel v_\perp (D^2_{pp}H\mathbf{v}^\parallel \cdot \mathbf{v}^\perp) + v_\perp^2 (D^2_{pp}H\mathbf{v}^\perp \cdot \mathbf{v}^\perp) \\
&\qquad \geq v_\perp^2 \left(\lambda\, |D_pH|^2 - C\,\right) - 2C|v_\parallel||v_\perp| + c\, v_\parallel^2 |D_pH| \\
&\qquad > v_\perp^2 \Big(\lambda\, \tau^2 - C\Big(1 + \frac{1}{\eta^2}\Big)\Big) + v_\parallel^2 (c\,\tau - C\eta^2\,).
\end{aligned}$$

Choosing first $\eta^2 < \frac{c\,\tau}{C}$, and then

$$\lambda > \frac{C}{\tau^2}\Big(1 + \frac{1}{\eta^2}\Big),$$

we obtain

$$(\lambda\, D_pH \otimes D_pH + D^2_{pp}H)v \cdot v \geq \alpha''|v|^2,$$

for some $\alpha'' > 0$, independent of (x, p).

General Case

In the general case, we have

$$(\lambda\, D_pH \otimes D_pH + D^2_{pp}H)v \cdot v \geq \gamma|v|^2,$$

where $\gamma := \min\{\alpha', \alpha''\}$.

Similar to the case above, we basically have $\phi(x, p) = e^{\lambda H(x,p)}$ is uniformly convex on the support of μ for λ large enough.

Hence, by doing the same steps as in [EG01] again, we finally get μ satisfies (2).

10. A one dimensional example of nonvanishing dissipation measure m

In this section we sketch a one dimensional example in which the dissipation measure m does not vanish. We assume that the zero level set of the Hamiltonian $H : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is the smooth curve in Figure 1, and that everywhere else in the plane (x, p) the signs of H are as shown in the picture. In addition, H can be constructed in such a way that $(D_xH, D_pH) \neq (0, 0)$ for every $(x, p) \in \{(x, p) \in \mathbb{T} \times \mathbb{R} : H(x, p) = 0\}$. That is, we require that no equilibrium point belongs to the zero level set of H. Consider now the piecewise continuous function $g : [0, 1] \to \mathbb{R}$, with $g(0) = g(1)$, as shown in Figure 2. Then, set

$$P := \int_0^1 g(x)\, dx,$$

and define

$$u(x, P) := -Px + \int_0^x g(y)\, dy.$$

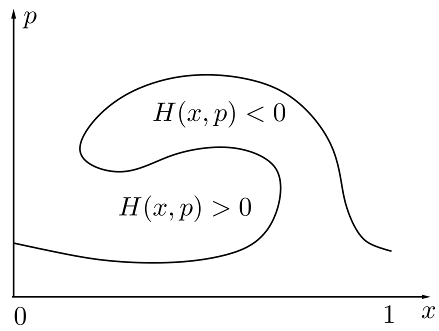

Figure 1. $\{H(x,p) = 0\}$.

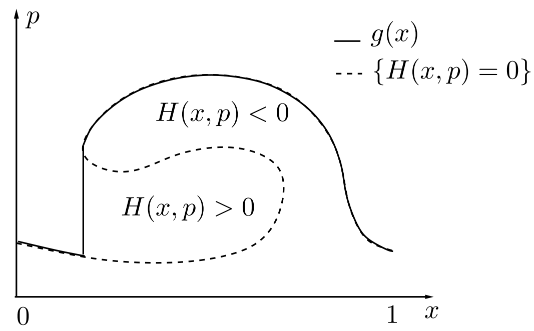

Figure 2. $g(x)$.

One can see that $u(\cdot, P)$ is the unique periodic viscosity solution of

$$H(x, P + D_x u(x, P)) = 0,$$

that is equation (1.5) with $\overline{H}(P) = 0$. Assume now that a Mather measure μ exists, satisfying property (1). Then, the support of μ has necessarily to be concentrated on the graph of g, and not on the whole level set $\{H = 0\}$, thus giving a contradiction.

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