Shadowing orbits for dissipative PDEs
( with G. Arioli )

(R) A few references

(A1) Generalities, the Kuramoto-Sivashinsky (KS) equation
(A2) The bifurcation diagram for the KS equation

(B1) A periodic orbit for the KS equation
(B2) Sketch of the proof
(B3) General framework for each step
(B4) Specifics for KS
(B5) The computer-assisted part


The methods apply in principle to general **dissipative evolution equations**

\[
\dot{u} + (-\Delta)^m u + H_\alpha(u, \nabla u, \ldots) = 0,
\]

for sufficiently simple domains and analytic nonlinearities \( H \).

Start with **stationary solutions** \( \dot{u} = 0 \) and rewrite the resulting equation as

\[
F_\alpha(u) = u, \quad \text{where} \quad F_\alpha(u) = -(-\Delta)^{-m} H_\alpha(u, \nabla u, \ldots),
\]

The idea is to exploit the compactness of \((-\Delta)^{-m}\) to obtain good finite dim approximations.

**Example.** The one-dimensional **Kuramoto-Sivashinsky (KS)** equation

\[
\partial_t u + 4\partial_x^4 u + \alpha(\partial_x^2 u + 2u\partial_x u) = 0, \quad t \geq 0, \quad x \in [0, \pi],
\]

with homogeneous Dirichlet boundary conditions.

A trivial solution is \( u = 0 \), for any value of \( \alpha \). It defines a line in the space of pairs \((\alpha, u)\) satisfying \( F_\alpha(u) = u \). Other solutions **bifurcate** off this line at \( \alpha = 4k^2 \), with \( k \) a positive integer. The resulting solution curves bifurcate again . . .

Determining the **bifurcation diagram** is simplified by the fact that many bifurcations involve the breaking of some symmetry.
Bifurcation diagram \((L^2 \text{ norm versus } \alpha)\) for the Kuramoto-Sivashinsky equation
Theorem. [G. Arioli, H.K.] For $0 \leq \alpha \leq 80$, the stationary KS equation exhibits eleven pitchfork bifurcations ($4, 16, 36, 64, P_1, P_2, iP_1, iP_2, iP_3^\pm, iP_4$), four intersection bifurcations ($I_1^\pm, I_2^\pm$), and eight fold bifurcations ($F^\pm, F_1^\pm, F_2^\pm, F_3^\pm$), connected by 44 smooth solution curves, as depicted below. These curves undergo no other bifurcations for $0 \leq \alpha \leq 80$.

Other results include bounds on the values of $\alpha$ for each bifurcation point, as well as the dimension of the unstable manifold and the $L^2$ norm of 30 selected solutions.
Non-stationary orbits. The goal is to solve initial value problems of the form

\[ \dot{u} = Lu + G(u), \quad u(0) = \nu, \]

where \( L \) is linear and “very negative”. Rewriting the equations as

\[ \partial_t [e^{-tL}u] = e^{-tL}G(u), \quad u(0) = \nu, \]

integrating both sides, and then multiplying by \( e^{tL} \), we get the integral equation

\[ u(t) = e^{tL}\nu + \int_0^t e^{(t-s)L}G(u(s)) \, ds. \] (**)

The flow \( \Phi \). After defining a suitable Banach space \( \mathcal{X} \) of admissible initial conditions \( \nu \), solve the equation by iteration, on a space of continuous curves \( u : [0, T] \rightarrow \mathcal{X} \). This yields the time-\( t \) maps \( \Phi_t(\nu) = u(t) \) for times up to \( T \).

Problem: Computer-estimates only work for small \( T > 0 \). Even composition of time-\( t \) maps gets soon out of control.

Way out: Shadowing of an approximate numerical orbit, using a sequence of boxes ...
Non-stationary solutions for KS. The most interesting are probably chaotic orbits, such as the ones found numerically in [F. Christiansen, P. Cvitanovic, V. Putkaradze, ’97] for $\alpha \approx 137$.

For “simplicity”, we focus on periodic orbits. Such orbits have also been constructed in [P. Zgliczyński, preprint '08], using different methods.

Recall KS:

$$\partial_t u = Lu - \alpha \partial_x (u^2), \quad L = -4\partial_x^4 - \alpha \partial_x^2.$$ 

Our “standard form” is obtained by splitting $L = L + L'$, with $L < 0$, and rewriting

$$\partial_t u = Lu + G(u), \quad G(u) = L' u - \alpha \partial_x (u^2).$$

Theorem. [G. Arioli, H.K.] The KS equation for $\alpha = 150$ has a hyperbolic periodic orbit with period $\tau = 0.00214688\ldots$. Some associated Poincaré map has a simple eigenvalue $\mu_1$ of modulus $|\mu_1| > 4.8$, and the remaining eigenvalues $\mu_2, \mu_3, \ldots$ lie in the disk $|\mu_n| < 0.69$.

Remarks.

- The derivative of the flow is estimated via the corresponding integral operator.
- The spaces used are far from optimal.
- The shadowing procedures uses $M = 4293$ rectangular boxes.
Sketch of the general procedure

The flow $\Phi$. Convert the integral equation (\textcolor{red}{\textcircled{\textbullet}}) to a fixed point equation for curves $u : [0, T] \rightarrow X$. Evaluating the solution $u$ at time $t$ defines the time-$t$ map, $\Phi_t(\nu) = u(t)$.

Here, $t$ can be replaced by an interval.

Local Poincaré map. Given a codimension one affine subspace $S$ transversal to the flow, define $P(\nu) = \Phi_{t(\nu)}(\nu)$, with $t(\nu)$ the smallest time $t > 0$ where $\Phi_t(\nu) \in S$.

A bound on $t(\nu)$ is an interval $[a, c]$ such that $\Phi_a(\nu)$ and $\Phi_c(\nu)$ lie on different sides of $S$. Then $\Phi_{[a,c]}(\nu)$ is an enclosure for $P(\nu)$.

The local Poincaré maps $P_j$. Using an approximate orbit $t \mapsto \bar{u}(t)$, choose $M$ milestones $\bar{u}_j$ along this orbit, and Poincaré section $S_j = \bar{u}_j + X_j$ transversal to $\bar{u}$. Define $P_j : S_{j-1} \rightarrow S_j$ as above.

Shadowing. In each section $S_j$ choose an appropriate box $B_j$ and check covering condition for $P_j(B_{j-1})$ and $B_j$. Here we use the derivative of $\Phi_t$.

In the periodic case ($j = 0$ is identified with $j = M$) this implies the existence of a fixed point for the full Poincaré map $\Psi = P_M \circ \ldots \circ P_2 \circ P_1$ and a closed orbit $u$ for the flow.

Linearized Poincaré maps. Let $u_j$ be the point where the orbit $u$ intersects $S_j$. Estimating the velocities $\dot{u}_j = Lu_j + G(u_j)$ gives bounds on the derivatives $DP_j(u_{j-1})$.

Hyperbolicity. Check cone conditions (linear analogue of covering conditions) for each $DP_j(u_{j-1})$. Then $D\Psi(u_0)$ satisfies a cone condition, and hyperbolicity follows.
**General framework**

Integration. Rewrite (\(\bullet\)) as fixed point problem for \(K_\nu(w) = w\) for \(w(t) = u(t) - e^{tL}\nu\), where

\[
(K_\nu(w))(t) = \int_0^t e^{(t-s)L}G(w(s) + e^{sL}\nu)\,ds, \quad 0 \leq t \leq T.
\]

Since the integrand can vary rapidly in \(t\) near \(t = 0\), partitioning \(J = [0,T]\) into \(n\) subintervals \(J_i = [t_{i-1}, t_i]\), with the partition being finer near \(t_0 = 0\), than near \(t_n = T\), and ...

Assuming the eigenfunction \(\{v_k\}\) of \(L\) span a dense subspace of \(\mathcal{X}\), define \(C(J,\mathcal{X})\) to be space of all functions \(w(t) = \sum_k w_k(t)v_k\) that have continuous coefficients \(w_k : J \to \mathbb{R}\), and a finite norm

\[
\|w\| = \max_i \sum_k \sup_{t \in J_i} \|w_k(t)v_k\|.
\]

The following is **specific to KS** (with \(\mathcal{X}\) defined later).

**Lemma 1.** \(K_\nu\) is a compact map on \(C(J,\mathcal{X})\), has a unique fixed point \(w\) for each \(\nu \in \mathcal{X}\), and the map \(\nu \mapsto w\) is of class \(C^1\). The flow \((t,\nu) \mapsto u(t)\) is of class \(C^1\) and compact, for \(0 < t \leq T\).
Shadowing. In the case of a single expanding direction, we can use the following

**Lemma 2.** Consider a Banach space $X = \mathbb{R} \oplus Z$, and let $V$ be the closed unit ball in $Z$. Let $F$ be a continuous and compact map

$$
[-1, 1] \times V \xrightarrow{F} \mathbb{R} \times V,
$$

$$
[-1, -\vartheta] \times V \xrightarrow{F} (-\infty, -1] \times V,
$$

$$
[\vartheta, 1] \times V \xrightarrow{F} [1, \infty) \times V,
$$

for some positive $\vartheta \leq 1$. Then $F$ has a fixed point in $[-\vartheta, \vartheta] \times V$.

Assume $X = \mathcal{Y} \oplus \mathcal{V} \oplus Z$, where $\mathcal{Y}$ and $\mathcal{V}$ are one-dimensional subspaces of $X$ (in our case roughly the unstable and velocity directions of the flow). Denote by $U$ and $V$ the closed unit balls in $\mathcal{Y}$ and $Z$, respectively.

**Definition.** A **section** (of $X$) is codimension one affine subspace of $X$. A **box** in a section $S$ is the image of $U \times V$ under a bi-continuous affine map $\psi : \mathcal{Y} \oplus Z \to S$.

**Definition.** Let $B_i = \psi_i(U \times V)$ and $B_j = \psi_j(U \times V)$ be boxes in two section $S_i$ and $S_j$, respectively. Given a map $f : B_i \to S_j$, we say that $B_i$ $f$-covers $B_j$ if the map $F : U \times V \to \mathcal{Y} \oplus Z$, defined by $F = \psi_j^{-1} \circ f \circ \psi_i$, satisfies the hypotheses of of Lemma 2, for some $\vartheta < 1$.

For simplicity, we identified here $\mathcal{Y}$ with $\mathbb{R}$, and $U$ with $[-1, 1]$.

**Corollary 3.** If for each $j$, the box $B_{j-1}$ $P_j$-covers $B_j$, then the Poincaré map $\Psi : S_0 \to S_0$, defined by $\Psi = P_M \circ \ldots \circ P_2 \circ P_1$, has a fixed point in $B_0$. 
Linearized Poincaré map at \( u_{j-1} \in S_{j-1} \).

\[
DP_j(u_{j-1})w = D\Phi_{t(u_{j-1})}(u_{j-1})w - \frac{\eta_j(D\Phi_{t(u_j)}(u_{j-1})w)}{\eta_j(\dot{u}_j)} \dot{u}_j.
\]

Here, \( \dot{u}_j = Lu_j + G(u_j) \) is the velocity at \( u_j = P_j(u_{j-1}) \).

And \( \eta_j \) is the linear functional that defines the hyperplane \( X_j \) at the section \( S_j = \tilde{u}_j + X_j \).

Consider now the points \( u_j \) where the periodic orbit intersects the Poincaré planes \( S_j \).

The low-frequency parts \( \ell_j = \mathbb{P}_L \dot{u}_j \) are estimated explicitly in our construction of the orbit.

To estimate the high-frequency parts \( h_j = \mathbb{P}_H \dot{u}_j \) use that \( \dot{u}_j = D\Phi_{t(u_{j-1})}(u_{j-1})\dot{u}_{j-1} \).

**Lemma 4.** Let \( k_j = \mathbb{P}_H D\Phi_{t(u_{j-1})}(u_{j-1})\ell_{j-1} \) and \( D_j = \mathbb{P}_H D\Phi_{t(u_{j-1})}(u_{j-1}) \mathbb{P}_H \). Then

\[
\|h_j\| \leq \|k_j\| + \|D_j\|\|h_{j-1}\|, \quad j = 1, 2, \ldots, M.
\]

In particular, if \( \|k_j\| \leq b \) and \( \|D_j\| \leq a < 1 \) for all \( j \), then \( \|h_j\| \leq (1 - a)^{-1}b \).
Hyperbolicity. In the case of a single expanding direction, we can use the following

Lemma 5. Let $A \neq 0$ be a bounded linear operator on a real Banach space $X = Y \oplus Z$, with $Y$ one-dimensional. Thus, if $y \in Y$ and $z \in Z$, we have a unique decomposition

$$A(y + z) = y' + z', \quad y' \in Y, \; z' \in Z.$$ 

Assume now that $A$ is compact, and that there exists positive real numbers $\beta < \alpha$, such that $\|z'\| \leq \beta \max\{\|y\|, \|z\|\}$, and such that $\|y'\| \geq \alpha\|y\|$ whenever $\|y\| \geq \|z\|$. Then $A$ has a simple eigenvalue $\lambda$ of modulus $|\lambda| \geq \alpha$, and no other eigenvalue of modulus $> \beta$.

Definition. Let $X = Y \oplus Z$, and let $\alpha > \beta$ be positive real numbers. Given two sections $\bar{u}_i + \psi_i(X_i)$ and $\bar{u}_j + X_j = \psi_j(X)$ of $X$, and a linear map $B : X_i \to X_j$, we say that $B$ satisfies the $(\alpha, \beta)$ cone condition, if $A = D\psi_j^{-1}BD\psi_i$ satisfies the hypotheses of Lemma 5.

Consider again the local Poincaré maps $P_j : S_{j-1} \to S_j$ described earlier. Denote by $u_j$ the intersection of the periodic orbit with the Poincaré plane $S_j$.

Corollary 6. If for each $j$, the derivative $DP_j(u_j)$ satisfies a $(\alpha_j, \beta_j)$ cone condition, then $D\Psi(u_0)$ has a simple eigenvalue $\mu_1$ of modulus $|\mu_1| \geq \prod_j \alpha_j$ and no other spectrum outside the disk $|\mu| \leq \prod_j \beta_j$. 
The KS equation \( \partial_t u = Lu + G(u) \). Recall that

\[ G(u) = L'u - \alpha \partial_x (u^2), \quad L + L' = -4\partial_x^4 - \alpha \partial_x^2, \]

with Dirichlet boundary conditions on \([0, \pi]\). The eigenvalues of \(-L\) and \(-L'\) are

\[
\lambda_k = \begin{cases} 
0, & \text{if } k \leq \kappa; \\
4k^4 - \alpha k^2 & \text{if } k > \kappa;
\end{cases} \quad \lambda'_k = \begin{cases} 
4k^4 - \alpha k^2, & \text{if } k \leq \kappa; \\
0, & \text{if } k > \kappa;
\end{cases}
\]

with eigenvectors \( v_k(x) = \sin(kx) \). Here, \( \kappa \geq \sqrt{\alpha}/2 \), so that \( \alpha k^2 - 4k^4 \leq 0 \) for \( k \geq \kappa \).

Function space used: \( \mathcal{X} = \mathcal{X}_1^o \) with \( \rho = 2^{-7} \).

Given \( \rho > 0 \), and a nonnegative integer \( K \), define

\( \mathcal{X}_K^o \) : Space of odd \( 2\pi \)-periodic real analytic functions on the strip \( |\text{Im} \, x| < \rho \),

\[
|u| = \sum_{k \geq K} |u_k| e^{\rho k} < \infty.
\]

\( \mathcal{X}_K^e \) : Analogous space of even \( 2\pi \)-periodic functions.
The computer-assisted proof uses a type

**Ball**: $S = (S.C, S.R) \in \text{Rep} \times \text{Radius}$.

representing intervals in $\mathbb{R}$, or balls in a Banach space $X$,

$$B(S) = (S.C) + (S.R)\mathbb{U}_\mathbb{R} , \quad B(S, X) = (S.R)\mathbb{U}_X .$$

where $\mathbb{U}_X = \{ x \in X : \|x\| \leq 1 \}$.

The representable sets in $\mathcal{X}_1^e$ are taken to be of the form

$$B(F) = \sum_{K=1}^{D} B(F.C(K)) \sin(K.) + \sum_{K=1}^{2D} B(F.E(K), \mathcal{X}_K^e) , \quad F \in \text{SFourier}.$$ 

The representable sets in $\mathcal{X}_0^e$ are defined analogously. Both are associated with data of type SFourier, which is an instantiation $(\text{FCoeff} \Rightarrow \text{Ball})$ of

**Fourier**: $F=(F.T, F.C, F.E)$, with $F.T$ encoding the type (even or odd, domain $\rho$), and

- $F.C$: array $[0..D]$ of FCoeff;
- $F.E$: array $[0..2*D]$ of FCoeff;

Implement bounds hierarchically, starting with simple and/or generic types, then for more complex types; first for basic operations, then for functions like $F_\alpha$. 
\( \mathcal{C}(J, \mathcal{X}_K^o) \) is the Banach space of all continuous functions \( w : J \to \mathcal{X}_K^o \) with …

\[
  w(t) = \sum_{k \geq K} w_k(t) v_k, \quad \|w\| = \max_i \|w\|_i, \quad \|w\|_i = \sum_{k \geq K} e^{\rho k} \max_{t \in J_i} |w_k(t)|.
\]

Simple representable sets for these spaces associated with data of type

ContFun: \( P=\langle P.C, P.E \rangle \), where

- \( P.C \): array \([0..P\text{-Deg}]\) of Ball;
- \( P.E \): array \([1..N\text{Err}]\) of Ball; (nonnegative)

\( \mathcal{B}(P.C) \): all polynomials of degree \( \leq P\text{-Deg} \), whose \( K\)-th coefficient belongs to \( \mathcal{B}(P.C(K), \mathbb{R}) \).

The polynomials on \( J = [0, T] \) are expanded about \( \frac{2}{3} T \).

\( \mathcal{B}(P.E, \mathcal{X}_K^o) \): all functions \( v \in \mathcal{C}(J, \mathcal{X}_K^o) \) such that \( \|v\|_I \leq P.E(I).R \) for all \( I \).

The representable sets for \( \mathcal{C}(J, \mathcal{X}_K^o) \) are associated with data of type TFourier, which is an instantiation (FCoeff \( \Rightarrow \) ContFun) of Fourier. In other words,

\[
  \mathcal{B}(F) = \sum_{K=0}^{D} \mathcal{B}(F.C(K)) \sin(K.) + \sum_{K=0}^{2D} \mathcal{B}(F.E(K), \mathcal{X}_K^o), \quad F \in \text{TFourier}.
\]

The representable sets for \( \mathcal{C}(J, \mathcal{X}_0^e) \) are defined analogously.

Now implement bounds Contr, DContr, ContrFix, DContrFix, Phi, DPhi, … on the maps \( K_\nu, \partial_\nu K_\nu, \ldots \).
To obtain decent error bounds for $\text{Contr}$, we decompose $K_{\nu}(w) = P(\nu, w) + Q(\nu, w)$, where $P$ is linear and $Q$ quadratic,

$$Q(\nu, w) = -\alpha \int_0^t e^{(t-s)L^+} \partial_x \left[ w + e^{sL} \nu \right]^2.$$

Then split $Q$ into terms $Q^{(n)}$ that are homogeneous of degree $n$ in $w$. After rewriting the result in terms of Fourier coefficients, we end up with integrals like

$$\left(Q^{(1+)}_{m}(\nu, w)\right)(t) = -\alpha m \sum_{k+\ell=m} \nu_k \int_0^t e^{-\lambda m(t-s)} e^{-\lambda k s} w_\ell(s) \, ds,$$

and use estimates like

$$\|Q^{(1+)}(\nu, w)\|_i \leq 2\alpha \|\nu\|_i \|w\|_i \sup_{k \in \mathcal{K}, \ell \in \mathcal{L}} \left[ \frac{k + \ell}{(\lambda_{k+\ell} - \lambda_k) + 2/t_i} \right] e^{-\lambda_k t_{i-1}}.$$

Here, $\mathcal{K}$ and $\mathcal{L}$ are the frequency ranges for $\nu$ and $w$, respectively. The $\sup$ is estimated by the program (beforehand), using monotonicity properties of $[\ldots]$.

$\text{ContrFix}$ first computes an approximate fixed point $w$ for $K_{\nu}$. Then it encloses $w$ in successively larger sets $B(F)$ until one of them is mapped into itself by $\text{Contr}$.

The same strategy is applied for $\text{DContr}$ and $\text{DContrFix}$.

Evaluating the result of $\text{ContrFix}$ at a specified time $t \in J$ yields a bound $\Phi$ on the flow $\Phi : (t, \nu) \mapsto u(t)$. 

As much as possible of the above is kept hidden at the higher “dynamical systems” level.

The package Boxes uses data types Vec to describe sets in $\mathcal{X}^p_1$.

$V(1..N)$ contains bounds on the first $N$ Fourier coefficients, and $V(N+1)$ is a bound on the norm of all “higher order” terms.

Other data types include LBasis, Frame, Box, TBox, ...

Roughly speaking, a Box represents a set $B = C + L(R_1) \times R_2 \times H$, with

- $C$: the center of the Box,
- $L(R_1)$: the image of $R_1 = [-1, 1] \times \{0\} \times [-1, 1]^{M-2}$ under an linear transformation $L$ on $\mathbb{R}^M$,
- $R_2$: a rectangle $R_2 = [-r_{M+1}, r_{M+1}] \times \ldots \times [-r_N, r_N]$,
- $H$: a “higher order” ball.

The zero-thickness direction of $L(R_1)$ corresponds to the Poincaré section.

A bound on the local Poincaré map is obtained by determining a time interval $T = [t - \varepsilon, t + \varepsilon]$ such that the flow-images of $B$ at the two times $t \pm \varepsilon$ lie on opposite sides of the Poincaré section (at the destination point).

The 4293 boxes used in our shadowing procedure have been determined numerically.

The box directions fall into 4 classes.

- **low**: The first 8 directions are roughly eigendirections of the return map (for the entire orbit).
- **mid-low**: The next 12 directions are $(I - P) \sin(k)$, for $k = 9, 10, \ldots, 20 = M$.
  
  Here, $P$ is an approximation to the “low” spectral projection.
- **mid-high**: Simply $\sin(k)$ for $k = 21, 22, \ldots, 40 = N$.
- **high**: All higher order modes ($k > N$).
Mapping a box \( B = b + L(R_1) \), where \( b = C + R_2 + H \).

Consider: a map \( f : B \to \text{somewhere} \), with \( f(0) = 0 \),
for every \( x \) a bound \( F(x) \) on \( Df(B)x \) (a convex set containing ...)
the “corners” \( b + w_i \) of \( B \), where \( w_1, w_2, \ldots, w_m \) are the corners of \( L(R_1) \).

Bound on \( f \) from bound on \( Df \): By convexity,

\[
f(x) = \int_0^1 Df(tx)x \in F(x), \quad \forall x \in B.
\]

Convex combination of corners: Every \( x \in B \) admits a unique representation

\[
x = \xi + \sum_i s_i w_i, \quad \xi \in b, \quad s_i \in [0, 1], \quad \sum_i s_i = 1.
\]

We have

\[
f(x) = \int_0^1 dt Df(tx)x = \sum_i s_i \int_0^1 dt Df(tx)(\xi + w_i) \in \sum_i s_i F(b + w_i).
\]

Notice: The bounds \( \{F(b + w_i)\}_{i=1}^m \) are sufficient to estimate \( f(x) \) for arbitrary \( x \in B \).