Equivariant Landau-Lifshitz equation of degree two

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1. Introduction

Long-time dynamics of the *Landau-Lifshitz* equation

$\vec{u}(t,x) : [0,\infty) \times \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$, $a_1 \geq 0$, $a_2 \in \mathbb{R}$

$$\vec{u}_t = (a_1 P^u + a_2 J^u)\Delta \vec{u}, \tag{1.1}$$

LL

$P^u \vec{v} := \vec{v} - \vec{u}(\vec{u}\cdot\vec{v})$: projection to $T_{\vec{u}}S^2 = \vec{u}^\perp$,
$J^u \vec{v} := \vec{u} \times \vec{v}$: rotation of $T_{\vec{u}}S^2$. $a := a_1 + ia_2$.

$$\begin{aligned} a = 1 &: \text{Harmonic map heat flow,} \\ a = i &: \text{Schrödinger map.} \end{aligned} \tag{1.2}$$

Space-time evolution of spin: A model of ferromagnetism.

$$\begin{aligned} &E(\vec{u}) := \int_{\mathbb{R}^2} |\nabla \vec{u}|^2 dx \quad \text{(energy)}, \\ &\partial_t E(\vec{u}) = -a_1 \|P^u \Delta \vec{u}\|^2_{L^2_x(\mathbb{R}^2)} \leq 0. \end{aligned} \tag{1.3}$$

We study global behavior of solutions around the simplest harmonic maps $\mathbb{R}^2 \cong \mathbb{C} \ni z \mapsto \mu z^m \in \overline{\mathbb{C}} \cong S^2$ ($\mu \in \mathbb{C}^\times$, $m \in \mathbb{N}$), under the symmetry (equivariance)

$$\vec{u}(x) = e^{m\theta\Omega}\vec{v}(r), \quad z = x_1 + ix_2 = re^{i\theta}, \quad \Omega := \vec{k}\times,$$
$$\vec{v}(r=0) = -\vec{k}, \quad \vec{v}(r=\infty) = \vec{k} := (0,0,1).$$

Let $\Sigma_m = \{\text{such } \vec{u}(x) \mid E(\vec{u}) < \infty\}$. μz^m is given by

$$\vec{u}(x) = e^{m\theta\Omega}\vec{h}[\mu], \qquad \vec{h}[\mu] := e^{\alpha\Omega}\vec{h}(r/s), \tag{1.4}$$

where $\mu := e^{i\alpha}s^{-m}$ and

$$\vec{h}(r) := \left(\frac{2}{r^m + r^{-m}},\ 0,\ \frac{r^m - r^{-m}}{r^m + r^{-m}}\right). \tag{1.5}$$

$\{e^{m\theta\Omega}\vec{h}[\mu]\}_{\mu\neq 0}$ are the ground states of Σ_m

$$\begin{aligned} \vec{\varphi}\in\Sigma_m \implies & E(\vec{\varphi})\geq 8m\pi, \\ & E(\vec{\varphi})=8m\pi \implies \vec{\varphi}\in\{e^{m\theta\Omega}\vec{h}[\mu]\}. \end{aligned} \tag{1.6}$$

Main Question:

Is the set $\{e^{m\theta\Omega}\vec{h}[\mu]\}$ asymptotically stable?

If so, how the solutions converge to them?

Special Feature:

Slow spatial decay of $\partial_\mu\vec{h}[\mu]$ (as $|x|\to\infty$) for lower m enhances the long-time interaction with the remainder.

Result:

t-asymptotic depends on x-decay of $\partial_\mu\vec{h}[\mu]$ (i.e. m).

Non-trivial asymptotic arises when $m=2$ for the heat flow.

2. Main results

Thm 1. *(higher m)*
Let $m \geq 3$, $a_1 \geq 0$, $a_2 \in \mathbb{R}$. *Then*
$\exists \delta > 0$, *for any* $\vec{u}(0) \in \Sigma_m$ *with* $E(\vec{u}(0)) \leq 8m\pi + \delta^2$,
$\exists ! \vec{u}(t,x)$ *solving* (1.1) *and*

$$\vec{u} \in C([0,\infty);\Sigma_m), \quad \nabla \vec{u} \in L^2_{loc}((0,\infty);L^\infty_x). \tag{2.1}$$

Moreover $\exists \mu \in \mathbb{C}^\times$, *s.t., as* $t \to \infty$,

$$\begin{aligned} &\|\vec{u}(t) - e^{m\theta\Omega}\vec{h}[\mu]\|_{L^\infty_x(\mathbb{R}^2)} \to 0, \\ &a_1 E(\vec{u}(t) - e^{m\theta\Omega}\vec{h}[\mu]) \to 0. \end{aligned} \tag{2.2}$$

Remark 1. This result for $m \geq 4$ had been proved by Gustafson-Kang-Tsai, Guan-Gustafson-Tsai ('09).

Thm 2. *($m = 2$, dissipative case)*
Let $m = 2$, $a_1 > 0$, $a_2 \in \mathbb{R}$. *Then*
$\exists \delta > 0$, *for any* $\vec{u}(0) \in \Sigma_m$ *with* $E(\vec{u}(0)) \le 8m\pi + \delta^2$,
$\exists ! \vec{u}(t,x)$ *solving* (1.1) *and*

$$\vec{u} \in C([0,\infty); \Sigma_m), \quad \nabla \vec{u} \in L^2_{loc}((0,\infty); L^\infty_x). \tag{2.3}$$

$\exists \mu : [0,\infty) \to \mathbb{C}^\times$, *s.t., as* $t \to \infty$,

$$\begin{aligned} &\|\vec{u}(t) - e^{m\theta\Omega}\vec{h}[\mu(t)]\|_{L^\infty_x(\mathbb{R}^2)} \to 0, \\ &E(\vec{u}(t) - e^{m\theta\Omega}\vec{h}[\mu(t)]) \to 0. \end{aligned} \tag{2.4}$$

$\mu(t)$ *is locally bounded by*

$$\int_0^T |\partial_t \log \mu(t)| dt \lesssim \delta \log(e + T/s(0)^2). \tag{2.5}$$

If in addition $a = 1$ *and* $v_2 \equiv 0$, *then* $\alpha \equiv 0$ *and*

$$(1 + o(1)) \log s(t) = \frac{2}{\pi} \int_1^{\sqrt{t}} \frac{v_1(0, r)}{r} dr + O_c(1), \qquad (2.6)$$

where $o(1) \to 0$, *and* $O_c(1)$ *converges. Moreover, each of the following (1)-(7) is realized by some initial data:*

(1) $s \to \exists s_\infty \in (0, \infty)$ (2) $s \to 0$ (3) $s \to \infty$
(4) $0 < \liminf s < \limsup s < \infty$
(5) $0 = \liminf s < \limsup s < \infty$
(6) $0 < \liminf s < \limsup s = \infty$
(7) $0 = \liminf s < \limsup s = \infty$

Each of (1)-(7) is stable for spatially local perturbation of the initial data (if the perturbation of v_1 *is integrable for* dr/r, $r \to \infty$*).*

3. Related known results

For heat flow ($a = 1$)

- Struwe: Global weak solution for finite energy data, with at most finite points of singularity, due to concentration of harmonic maps. Uniqueness proved by Freire.
- Chang-Ding-Ye: Concentration actually occurs for $m = 1$ and $v_2 \equiv 0$ on a disk. (Guan-Gustafson-Tsai on $\mathbb{R}^2$)
- Angenent-Hulshof: Concentration as $t \to \infty$ for $m = 2$ and $v_2 \equiv 0$ on a disk.

For Landau-Lifshitz ($a_1 > 0$, $a_2 \neq 0$)

- Guo-Hong : Global weak solutions of Struwe-type.

For Schrödinger map ($a = i$)

- Chang-Shatah-Uhlenbeck: Global wellposedness under the symmetry for $m \geq 0$ and $E \ll 1$.
- Bejenaru-Ionescu-Kenig-Tataru:
  Globally wellposed without symmetry for $E \ll 1$.

For wave map ($u_{tt} = P^u \Delta u$)

- Sterbenz-Tataru: Singularity is concentration of Lorentz transformed harmonic maps. Globally wellposed for $E < 8\pi$.
- Krieger-Schlag-Tataru, Rodnianski-Sterbenz, Raphael-Rodnianski: Concentration for $m \geq 1$, $v_2 \equiv 0$.

For semilinear heat ($u_t = \Delta u + |u|^\alpha u$, $u : \mathbb{R}^{1+d} \to \mathbb{R}$)

- Poláčik-Yanagida: construct solutions with asymptotes (1)-(7) for $\alpha \geq 4/[d - 4 - 2\sqrt{d-1}]$ ($d \geq 11$).

4. Standard argument, Difficulty and Ideas

Decompose the solution $\vec{v}(t,r) = e^{-m\theta\Omega}\vec{u}(t,x)$

$$\vec{v}(t) = \vec{h}[\mu(t)] + \check{v}(t) \tag{4.1}$$

such that

(1) $\check{v}(t)$ decays by dispersion/dissipation.
(2) $\mu(t)$ converges due to integrability of nonlinear terms.

$$\text{span}\left\{\frac{\partial \vec{h}}{\partial \mu}\right\} = \text{Kernel of the linearized op. for } \check{v} \tag{4.2}$$

Let $\dot{\mu}(t)$ = the component of the nonlinear interaction in the $\partial_\mu \vec{h}[\mu]$ direction. Then

(1) $\check{v}$ consists of continuous spectrum ($\Longrightarrow$ dispersion)
(2) $\dot{\mu}$ is at least quadratic in $\check{v}$ (integrable if $\check{v} \in L^2_t$)

4.1. **Difficulty.** Compatibility of the decomposition and the dispersive/dissipative estimate.

Time decay must be with $\underline{\text{weaker spatial decay}}$ than L^2_x

$\Uparrow$

can be destroyed by the orthogonal decomposition

if the eigenfunction decays slowly (i.e. lower m)

The decay order is determined by the scale invariance:

$$\vec{u}(t,x) : \text{sol.} \implies \vec{u}_\lambda(t,x) := \vec{u}(\lambda^2 t, \lambda x) : \text{sol.}, \\ E(\vec{u}_\lambda(0)) = E(\vec{u}(0)). \tag{4.3}$$

Decay estimate must be invariant for this, and we want L^2_t for $\check{v}$. Then the spatial order is given by

$$\check{v}/r \in L^2_t L^\infty_x. \tag{4.4}$$

The kernel $\partial_\mu \vec{h}[\mu] = h_1/\mu \cdot e^{\alpha\Omega}(-h_3, 0, h_1)(r/s)$,

$$h_1 = \frac{2}{r^m + r^{-m}} = O(r^{-m}) \quad (r \to \infty), \tag{4.5}$$

whose orthogonal projection preserves the above space rL_x^∞

$$\text{if and only if } rh_1(r) \in L_x^1(\mathbb{R}^2) \quad (\text{i.e. } m > 3). \tag{4.6}$$

4.2. **Idea 1.** Localize the orthogonality:
Fix $\varphi(r) \in C_0^\infty(0, \infty)$ s.t. $(h_1|\varphi) = 1$, and decompose

$$L_x^2 = (h_1) \oplus \varphi^\perp \tag{4.7}$$

(with the rescaling by $s(t)$). Then
Merit: it preserves decay estimates.
Demerit: it is not preserved by the linearized evolution.
Hence $\dot{\mu}$ gets a linear term in $\check{v} \notin L_t^1$.

4.3. **Hasimoto transform.**
$\partial_{\bar{z}}\vec{v}$ behaves "more linearly" in some orthonormal frame, satisfying a NLS-like equation and time decay.

4.4. **Idea 2.** Partial integrations in space and in time, of the term $O(\check{v})$ in the μ equation, using the equation of $\partial_{\bar{z}}\vec{v}$.
This gives the leading term of $s(t)$ if $m = 2$ and $\alpha \equiv 0$.
Problem: a term with $\dot{\alpha}$ cannot be integrated if $m = 2$.

4.5. **Idea 3.** A priori bound on $\int |\partial_t \log \mu| dt$ on dyadic time intervals $[T_j, 2T_j]$, $T_j = 2^j s(0)^2$, using weighted L^p estimates for the linearized operator around $h[\mu(T_j)]$.
The slow move $s(t) = O(t^{C\delta})$ does not affect much the main interaction around $r^2 \sim t$.
Demerit: The weighted estimates require $a_1 > 0$.

5. More details

5.1. Decomposition and localized orthogonality. Let $\mathbf{f}(t,r) \in \mathbb{C}^3 \perp \vec{h}[\mu]$ be the orthonormal frame

$$\mathbf{f} := e^{\alpha\Omega}(h^s \times \vec{j} + i\vec{j}), \quad \vec{j} := (0,1,0), \tag{5.1}$$

where $h^s(r) = h(r/s)$, $\mu = e^{i\alpha}s^{-m}$.

Let $\widetilde{v}(t,r) := \vec{v} \cdot \mathbf{f} \in \mathbb{C}$ be the component of $\vec{v}$ in $\vec{h}[\mu]^\perp$.

$$v = [1 + O(|\widetilde{v}|^2)]\vec{h}[\mu] + \mathrm{Re}(\widetilde{v}\overline{\mathbf{f}}), \tag{5.2}$$

i.e. $\widetilde{v} \approx \check{v}$. We choose μ by the localized orthogonality

$$0 = (\widetilde{v}|\varphi^{\not s}) := \int_{\mathbb{R}^2} \widetilde{v}(x)\overline{\varphi^{\not s}}(x)dx, \tag{5.3}$$

where $\varphi^{\not s}(r) = s^{-2}\varphi(r/s)$ and $\varphi(r) \in C_0^\infty(0,\infty)$ is fixed such that $(\varphi|h_1) = 1$.

5.2. **Generalized Hasimoto transform.** (Chang-Shatah-Uhlenbeck) Let $\mathbf{e}(t,r) \in T_v \oplus iT_v$ be the orthonormal frame $i\mathbf{e} = v \times \mathbf{e}$, solving

$$P^v \mathbf{e}_r = 0, \quad \mathbf{e}(r \to \infty) = (-1, i, 0). \tag{5.4}$$

Define $q(t,r), \nu(t,r) \in \mathbb{C}$, and $\vec{w}(t,r) \in \vec{v}^{\perp}$ by

$$\vec{w} = \vec{v}_r - \frac{m}{r} P^v \vec{k}, \quad q = \vec{w} \cdot \mathbf{e}, \quad \nu = P^v \vec{k} \cdot \mathbf{e}. \tag{5.5}$$

Then $E(u) = 8\pi m + \|q\|_{L^2}^2$ and from the v equation

$$\begin{aligned} & q_t + iSq = -aL_v L_v^* q, \quad L_v := \partial_r + \frac{mv_3}{r}, \\ & S = \mathrm{Re} \int_{\infty}^{r} (q + \frac{m}{r}\nu) \overline{iaL_v^* q} dr. \end{aligned} \tag{5.6}$$

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Decay estimate on q: The energy identity

$$\|q(T)\|^2_{L^2_x} + 2a_1 \|L^*_v q\|^2_{L^2_{t,x}(0<t<T)} = \|q(0)\|^2_{L^2_x} \qquad (5.7)$$

gives $q \in L^2_t L^\infty_x$ (via Sobolev) if $a_1 > 0$. If $a_1 = 0$, we use the Strichartz estimate in $L^2_t L^\infty_x$ for $e^{itL_h L^*_h}$.

From q to $\check{v}$: Let $L^s := L_{h^s} = \partial_r + m h^s_3 / r$ then

$$L^s \widetilde{v} = \vec{w} \cdot \mathbf{f} + O(\delta \widetilde{v}/r), \qquad (5.8)$$

where $|\check{v}| \lesssim |\widetilde{v}| + |\widetilde{v}|^2$ and $|\vec{w}| = |q|$. We can invert L^s by integration, using the localized orthogonality $(\widetilde{v}|\varphi^{\not{s}}) = 0$. Thus we get

$$\|\widetilde{v}/r\|_{L^2_x} \lesssim \|\varphi\|_{L^1_x} \|q\|_{L^2_x}, \quad \|\widetilde{v}/r\|_{L^\infty_x} \lesssim \|r\varphi\|_{L^1_x} \|q\|_{L^\infty_x}.$$

If φ is replaced with $h_1 = O(r^{-m})$, then these require $m > 2$ and $m > 3$, respectively.

5.3. **Equation for $\mu(t)$.** From $\partial_t(\widetilde{v}|\varphi^{\not s}) = 0$ we get

$$\dot{\mu}/\mu = -ae^{i\widetilde{\alpha}}(L_v^* q|\varphi^{\not s}) + L_t^1, \quad (\widetilde{\alpha}_t = \alpha_t + L_t^1). \tag{5.9}$$

$(L_v^* q|\varphi^{\not s})$ remains because $(\widetilde{v}|\varphi^{\not s}) = 0$ is not preserved by the equation. $e^{i\widetilde{\alpha}}$ is due to the rotation of $\mathbf{f}$.

Partial integrations: Using the equation of q,

$$\begin{aligned} \dot{\mu}/\mu &= -ae^{i\widetilde{\alpha}}(L_v L_v^* q|\psi^s/s) + L_t^1 \\ &= e^{i\widetilde{\alpha}}(q_t|\psi^s/s) + L_t^1, \end{aligned} \tag{5.10}$$

for some $\psi(r) = O(r^{1-m})$ obtained by integration in r. Integration in t gives us

$$\begin{aligned} &\partial_t \left[\log\mu - e^{i\widetilde{\alpha}}(q|\psi^s/s)\right] \\ &= -is\widetilde{\alpha}_t e^{i\widetilde{\alpha}}(q|\psi^{\not s}) - s_t e^{i\widetilde{\alpha}}(q|(r\partial_r + 1)\psi^{\not s}) + L_t^1. \end{aligned} \tag{5.11}$$

If either $m \geq 3$ or $m = 2$ and $\alpha \equiv 0$, we can prove the whole RHS is in L^1_t, i.e.

$$\int_0^\infty |\partial_t[\log \mu - e^{i\widetilde{\alpha}}(q|\psi^s/s)]|dt \lesssim \delta. \tag{5.12}$$

The leading term is bounded if $m \geq 3$.
If $m = 2$, $a = 1$, and $v_2 \equiv 0$, then we can derive

$$[(q|\psi^s/s)]_0^t \approx \frac{1}{\pi}\int_{s(0)}^{\sqrt{t}} q(0,r)dr = \frac{1}{\pi}\int_{s(0)}^{\sqrt{t}} \frac{2v_1(0,r)}{r}dr,$$

approximating q by the heat equation.
But if $m = 2$ and $\alpha \not\equiv 0$, then the term $\widetilde{\alpha}_t e^{i\widetilde{\alpha}}(q|\psi^{\not s})$ in $\dot{\mu}/\mu$ cannot be bounded, since $(q|\varphi^{\not s})$ is unbounded and we have only $s\widetilde{\alpha}_t \in L^2_t$.

5.4. A priori bound on μ. If $m = 2$ and $a_1 > 0 \neq a_2$, then instead of the parital integrations, we estimate

$$\int_0^T |\dot{\mu}/\mu| dt = \int_0^T |(L^{s*} q|\varphi^{\not{s}})| dt + O(\delta), \tag{5.13}$$

where $L^s = \partial_r + m h_3^s / r$,

$$|(L^{s*} q|\varphi^{\not{s}})| = |(q|L^s \varphi^{\not{s}})| \lesssim \min(\|q\|_{L_x^2}/s^2, \|q/r\|_{L_x^\infty}).$$

On each interval $[T_j, T_{j+1}]$, $T_j = 2^j s(0)^2$, we use weighted estimates for $H^j := L^{s(T_j)} L^{s(T_j)*}$ such as

$$\|e^{-atH^j}\varphi\|_{L_t^1(T_1,T_2;rL_x^\infty)} \lesssim \|\min(r^2/T_1, 1, \sqrt{T_2}/r)\varphi\|_{rL_x^1},$$

and the Duhamel formula

$$q(t) = e^{a(T_j - t)H^j} q(T_j) + \int_{T_j}^t e^{a(t'-t)H^j}(q_t + aH^j q)(t') dt'.$$

The error term from the moving potential

$$[H^j - L^s L^{s*}]q = -2mar^{-2}[h_3^{s(T_j)} - h_3^{s(t)}]q \tag{5.14}$$

is bounded on $[T_j, T_{j+1}]$ by

$$\|\log(s/s(T_j))\|_{L_t^\infty}\|q/r\|_{L_x^\infty} \lesssim \|\dot{\mu}/\mu\|_{L_t^1}\|q/r\|_{L_x^\infty}. \tag{5.15}$$

Thus we obtain on each interval $[T_j, T_{j+1}]$,

$$\begin{aligned} \|\dot{\mu}/\mu\|_{L_t^1} &\lesssim \|q/r\|_{L_t^1 L_x^\infty} + \delta \\ &\lesssim \|\dot{\mu}/\mu\|_{L_t^1}\|q/r\|_{L_t^1 L_x^\infty} + \delta, \end{aligned} \tag{5.16}$$

and hence $|\log \mu(t)| \lesssim \delta j$ for $t \sim 2^j s(0)^2$.

We can improve the bound to

$$\|\dot{\mu}/\mu\|_{L_t^1(T_j, T_{j+1})} \lesssim \delta + \|\min(r^2/T_j, \sqrt{T}_j/r)q(0)\|_{rL_x^1}.$$

But the weighted estimates do not hold in the Schrödinger case $a = i$.