

Stochastic Completeness of Graphs

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Setting

Let $G = (V, E)$ denote an infinite, locally finite, connected graph.

That is, V is a (countably) infinite set of vertices and E is the set of edges of G . We will write $x \sim y$ if x and y form an edge and say that x and y are **neighbors**.

Locally finite means that $\forall x \in V$

$$m(x) := |\{y \mid y \sim x\}| < \infty$$

that is, the **degree** or **valence** is finite at each vertex.

Connected means that $\forall x, y \in V$ there exists a sequence $\{x_i\}_{i=1}^n$ where $x_i \in V$ such that $x \sim x_1 \sim x_2 \sim \dots \sim x_n \sim y$ which we call a **path** connecting x and y .

Let $d(x, y)$ denote the number of edges in the shortest path connecting x and y with $d(x, x) = 0$.

Let $C_0(V) = \{f : V \rightarrow \mathbb{R} \mid |\text{supp } f| < \infty\}$ denote the space of finitely supported functions on G and

$$\ell^2(V) = \{f : V \rightarrow \mathbb{R} \mid \sum_{x \in V} f^2(x) < \infty\}$$

denote the Hilbert space of square summable functions on G with inner product given by

$$\langle f, g \rangle = \sum_{x \in V} f(x)g(x).$$

If $f : V \rightarrow \mathbb{R}$ is a real-valued function on G , then $\Delta f : V \rightarrow \mathbb{R}$, the **Laplacian on G** is defined by

$$\Delta f(x) = \sum_{y \sim x} (f(x) - f(y)) = m(x)f(x) - \sum_{y \sim x} f(y).$$

Basic Facts about the Laplacian

- (1) Δ with domain $C_0(V)$ is essentially self-adjoint. That is, Δ extends uniquely to a self-adjoint operator with domain

$$D(\Delta) = \{f \in \ell^2(V) \mid \Delta f \in \ell^2(V)\}.$$

- (2) Δ is non-negative. That is, $\langle \Delta f, f \rangle \geq 0$ for $f \in C_0(V)$.
- (3) Δ is bounded $\iff m(x) \leq K$ for all $x \in V$.

Stochastic Completeness

For $t \in \mathbb{R}_+$ and $x, y \in V$ let $p_t(x, y)$ denote the **heat kernel** on G . That is, if $u_0 : V \rightarrow \mathbb{R}$ is any bounded, non-negative function on G , then

$$u(x, t) = \sum_{y \in V} p_t(x, y) u_0(y)$$

denotes the smallest positive bounded solution to the heat equation with initial condition u_0 :

$$\left(\Delta + \frac{\partial}{\partial t} \right) u(x, t) = 0 \text{ for } x \in V, t > 0$$

$$\lim_{t \rightarrow 0^+} u(x, t) = u_0(x).$$

Question: Are bounded solutions of the heat equation uniquely determined by initial data?

A Characterization

Theorem

The following properties are equivalent:

- (1) Bounded solutions of the heat equation are uniquely determined by initial data.*
- (2) $\sum_{y \in V} p_t(x, y) = 1$ for some (equivalently, all) $x \in V$ and $t > 0$.*
- (3) If $v > 0$ satisfies $\Delta v = \lambda v$ for $\lambda < 0$, then v is not bounded.*

A graph satisfying any of the conditions above is called **stochastically complete**. Otherwise, it will be called **stochastically incomplete**. Functions satisfying $\Delta v = \lambda v$ will be called λ -**harmonic**.

Note: The characterization above is due to Has'minskii (1960) and Feller (1957) for Euclidean space and Feller (1957) and Reuter (1957) for discrete spaces.

Question: What geometric properties imply stochastic completeness or incompleteness?

To motivate our results on graphs let us mention a few results for geodesically complete Riemannian manifolds.

- (1) (Azencott, 1974) If M is a simply connected, negatively curved Riemannian manifold and $K(r)$ denotes the infimum of the absolute values of sectional curvatures at distance r , then

$$K(r) \geq r^{2+\epsilon} \text{ for } \epsilon > 0$$

implies stochastic incompleteness. On the other hand, if the sectional curvature on such a manifold is bounded, then M is stochastically complete.

- (2) (Yau, 1978) Ricci curvature bounded from below implies stochastic completeness.
- (3) (Grigor'yan, 1986) If $\int_0^\infty \frac{r}{\ln V(r)} dr = \infty$, where $V(r)$ denotes the volume of a geodesic ball, then M is stochastically complete.

Completeness Criteria

For graphs, our starting point is the following result:

Theorem

(Dodziuk-Mathai, 2006) *If $m(x) \leq K$, then G is stochastically complete.*

To state an improvement, we need to introduce some notation. Fix a vertex $x_0 \in V$ and let $r(x) = d(x, x_0)$. Furthermore, let $m_{\pm}(x) = |\{y \mid r(y) = r(x) \pm 1\}|$, that is, the number of neighbors of x that are one step further or closer to x_0 than is x . Let $K_+(r) = \max_{x \in S_r} m_+(x)$ where $S_r = \{x \mid r(x) = r\}$.

Theorem

(W, 2009) *If $\sum_{r=0}^{\infty} \frac{1}{K_+(r)} = \infty$, then G is stochastically complete.*

Stochastically Incomplete Graphs

For trees, that is, for simply connected graphs which we denote by T , there is a complementary result as follows: let $k_+(r) = \min_{x \in S_r} m_+(x)$.

Theorem

(W, 2009) *If $\sum_{r=0}^{\infty} \frac{1}{k_+(r)} < \infty$, then T is stochastically incomplete.*

Hence trees whose spheres grow factorially yield examples of stochastically incomplete graphs. These examples can be seen as analogue for the examples of Azencott in the Riemannian manifold case mentioned above.

Spherically Symmetric Graphs

In the manifold case, Grigor'yan's result that $\int^{\infty} \frac{r}{\ln V(r)} dr = \infty$ implies stochastic completeness is optimal. It can be shown to be sharp by considering the case of spherically symmetric manifolds. However, we will show that in the graph case no analogous statement can hold if one uses the usual notions of distance and volume. In fact, by considering the case of spherically symmetric graphs, we will show that there exist stochastically incomplete graphs of polynomial volume growth.

We will say that a graph is **spherically symmetric** provided that it contains a vertex x_0 such that $m_{\pm}(x)$ defined with respect to x_0 as above depend only on the distance from x_0 . We then denote the common values of $m_{\pm}(x)$ for $x \in S_r$ by $k_{\pm}(r)$ and denote such graphs by $G_{k_{\pm}}$.

We let $S(r)$ denote the number of vertices in S_r ,
 $V(r) = \sum_{i=0}^r S(i)$, the number of vertices in B_r , the ball of radius r about x_0 , and $\partial B(r) = |\{y \sim x \mid x \in S_r, y \notin B_r\}|$, the number of edges connecting B_r to its complement.

With these notations we can state:

Theorem

(W, 2009) $G_{k\pm}$ is stochastically complete $\iff \sum_{r=0}^{\infty} \frac{V(r)}{\partial B(r)} = \infty$.

This theorem can be seen as an analogue for the case of spherically symmetric (or model) manifolds for which stochastic completeness is equivalent to $\int^{\infty} \frac{V(r)}{S(r)} dr = \infty$.

Examples:

- (1) For spherically symmetric trees T_{k_+} , that is, when $k_-(r) = 1$ and there are no connections between vertices on the same sphere, so that T_{k_+} is completely determined by k_+ , it follows that T_{k_+} is stochastically complete if and only if $\sum_{r=0}^{\infty} \frac{1}{k_+(r)} = \infty$. Note that adding edges along a sphere of a given radius has no effect on the criterion.
- (2) Consider the following construction. Choose $S(r) \in \mathbb{N}$ arbitrarily with $S(0) = 1$. Then connect each vertex in S_r to every vertex in S_{r+1} for all r . Denote such graphs by G_S . It follows that $k_{\pm}(r) = S(r \pm 1)$ and, from the above, that G_S is stochastically complete if and only if $\sum_{r=0}^{\infty} \frac{\sum_{i=0}^r S(i)}{S(r)S(r+1)} = \infty$. Hence, if $S(r) \geq r^{2+\epsilon}$ for $\epsilon > 0$, then G_S is stochastically incomplete.

Connections to Spectral Properties

Recall, that the Laplacian is a positive, essentially self-adjoint operator on $\ell^2(V)$. We denote the spectrum of Δ by $\sigma(\Delta)$ and the bottom of the spectrum by $\lambda_0(\Delta)$. The spectrum can be decomposed as a disjoint union:

$$\sigma(\Delta) = \sigma_{disc}(\Delta) \cup \sigma_{ess}(\Delta)$$

where $\sigma_{disc}(\Delta)$, the **discrete spectrum**, consists of isolated eigenvalues of finite multiplicity and $\sigma_{ess}(\Delta)$, the **essential spectrum**, is its complement in $\sigma(\Delta)$.

For spherically symmetric graphs, there is a relation between stochastic incompleteness and spectral properties which can be stated as follows:

Theorem

(Keller-W, 2010) If $G_{k_{\pm}}$ satisfies $\sum_{r=0}^{\infty} \frac{V(r)}{\partial B(r)} = a$, then

$$\lambda_0(\Delta) \geq \frac{1}{a} \text{ and } \sigma_{\text{ess}}(\Delta) = \emptyset.$$

Therefore, for stochastically incomplete spherically symmetric graphs the bottom of the spectrum is positive and the spectrum is discrete. We note that neither implication reverses, even for spherically symmetric graphs. For example, it is known that regular trees have positive bottom of the spectrum but they are clearly stochastically complete by the result of Dodziuk and Mathai.

Furthermore, by a result of Urakawa (1999), for any tree for which $m^c(r) \rightarrow \infty$, where $m^c(r) = \inf_{x \in B_r^c}$ for $B_r^c = G \setminus B_r$, the spectrum of Laplacian is discrete, whereas for stochastic incompleteness a certain growth is required as mentioned above.

Furthermore, the result is not true for graphs which are not spherically symmetric. This can be seen as follows: if one starts with a stochastically incomplete graph, then attaching a graph at a single vertex has no effect on the stochastic incompleteness but can drive the bottom of the spectrum down to zero and can induce essential spectrum.

For example, attaching a single path to infinity at any vertex of a stochastically incomplete graph drives down the bottom of the spectrum, since, for any finitely supported function f , $\lambda_0(\Delta) \leq \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle}$, which implies that

$$\lambda_0(\Delta) \leq \frac{\partial D}{|D|}$$

where $\partial D = |\{y \sim x \mid x \in D \text{ and } y \notin D\}|$ for $D \subset G$ finite.

That one can add essential spectrum by attaching a graph can be seen from the following theorem:

Theorem

(Keller, 2010) For any graph G with $\tilde{\alpha}_\infty > 0$,

$$\sigma_{\text{ess}}(\Delta) = \emptyset \iff m^c(r) \rightarrow \infty.$$

Here, $\tilde{\alpha}_\infty$ is a Cheeger constant at infinity which is defined by taking

$$\tilde{\alpha}_r = \inf_{\substack{D \subset B_r^c \\ |D| < \infty}} \frac{\partial D}{m(D)}$$

where ∂D is defined above and $m(D) = \sum_{x \in D} m(x)$ and letting $\tilde{\alpha}_\infty = \lim_{r \rightarrow \infty} \tilde{\alpha}_r$. Therefore, by attaching a suitable graph such as a regular tree of degree 3, one can induce essential spectrum while not effecting the stochastic incompleteness.

On the other hand, by results of Fujiwara (1995, 1996), it follows that for graphs with subexponential volume growth $\tilde{\alpha}_\infty = 0$ so that Keller's result mentioned above does not apply in our generality.

Our result also shows that no analogue of Brook's theorem concerning the bottom of the (essential) spectrum holds for our Laplacian with the usual notions of distance and volume.

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