On the Boltzmann limit for a Fermi gas in a random medium with dynamical Hartree-Fock interactions

Thomas Chen

Department of Mathematics
University of Texas at Austin

Classical and Random Dynamics in Mathematical Physics

University of Texas at Austin, 2010

Motivating Problem

Extremely difficult: Interacting Fermi gas on $\Lambda_L := [-L, L]^d \cap \mathbb{Z}^d$

$$H_\lambda \ := \ \int dp\, E(p)\, n_p \ + \ \lambda \sum_{x,y\in\Lambda_L} n_x\, v(x-y)\, n_y \ .$$

Conjecture: Let $t = \frac{T}{\lambda^2}$, and ρ_0 free Gibbs state. Then,

$$F_T(p) := \lim_{\lambda\to 0} \lim_{L\to\infty} \rho_0(\, e^{iT/\lambda^2\, H_\lambda}\, n_p\, e^{-iT/\lambda^2\, H_\lambda}\,)$$

exists and satisfies the *Boltzmann-Uhlenbeck-Uehling equation*

$$\begin{aligned}
&\partial_T F_T(p) \\
&\quad = \quad -4\,\pi \int dp_1\, dp_2\, dq_1\, dq_2\, \delta(\, p - p_1\,) \Big| \widehat{v}(p_1 - q_1) - \widehat{v}(p_1 - q_2) \Big|^2 \\
&\qquad\qquad \delta(\, p_1 + p_2 - q_1 - q_2\,)\ \delta(\, E(p_1) + E(p_2) - E(q_1) - E(q_2)\,) \\
&\qquad\qquad \Big[\, F_T(p_1) F_T(p_2) \widetilde{F}_T(q_1) \widetilde{F}_T(q_2) \ - \ F_T(q_1) F_T(q_2) \widetilde{F}_T(p_1) \widetilde{F}_T(p_2) \Big] ,
\end{aligned}$$

where $\widetilde{F}_T(p) := 1 - F_T(p)$.

Some work towards Boltzmann-Uhlenbeck-Uehling limit from interacting Fermi gas.

- Benedetto, Castella, Esposito, Pulvirenti [04]
  ($\exists$ subseries in Feynman graph expansion yielding BUU. No control on errors)
- Hugenholtz [83]
  (physical argumentation motivating BUU limit)
- Ho-Landau [97]
  (proof to order $O(\lambda^2)$)
- Erdös-Salmhofer-Yau [04]
  (quasifreeness of correlations implies BUU)
- Spohn [06] (survey article)
- C-<u>Sasaki</u>, '08
  (ok to order $O(\lambda^3)$, probably ok to $O(\lambda^4)$, unpublished)

Quantum dynamics of fermion gases in random media

Consider electron gas in a medium containing randomly distributed impurities (e.g. semiconductor).

Another interpretation: Randomness $\sim$ simplification of particle interaction with all other fermions in BUU problem.

Question:

- How do the Pauli principle and manybody interactions modify the transport properties ?
- Dynamics of the momentum distribution ?

In this talk:

- Joint with Itaru Sasaki (Shinshu U.) [JSP, 08]:
  Ideal Fermi gas in random medium.
- Joint work with Igor Rodnianski (Princeton):
  Mean field interacting Fermi gas in random medium.

Prove that kinetic scaling limit of momentum distribution function is determined by solution of a Boltzmann equation.

Related Works

(Derivation of dynamical Hartree-Fock equations)

- Bardos-Ducomet-Golse-Gottlieb-Mauser, [07].
- Bove-Da Prato-Fano [76]

Basics

Box $\Lambda_L \subset \mathbb{Z}^d$ of size $L \gg 1$, dual lattice $\Lambda_L^* = \Lambda_L / L \subset \mathbb{T}^d$.

Hilbert space of Fermi field $\mathfrak{F} = \oplus_{n \geq 0} \mathfrak{F}_n$

$\mathfrak{F}_n$ = completely antisymmetric ℓ^2-functions in n variables.

Creation operators $a_p^+ : \mathfrak{F}_n \to \mathfrak{F}_{n+1}$ and annihilation operators $a_q : \mathfrak{F}_n \to \mathfrak{F}_{n-1}$, satisfying canonical anticommutation relations

$$a_p^+ \, a_q + a_q \, a_p^+ = \delta(p - q) := \begin{cases} L^d & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases}$$

Remark: The operators

$$n_p := \frac{1}{L^d} \, a_p^+ \, a_p \quad \text{resp.} \quad n_x := a_x^+ \, a_x$$

count the # of electrons (0 or 1) with momentum p or position x.

1. Ideal Fermi gas in random medium

Random Hamiltonian for fermion field

$$H_\omega := T + \eta V_\omega$$

Kinetic energy operator with 1-electron kinetic energy $E(p)$

$$T = \sum_{p \in \Lambda_L^*} E(p)\, n_p$$

Random potential, $\{\omega_x\}$ i.i.d., centered, normalized, Gaussian

$$V_\omega := \sum_{x \in \Lambda_L} \omega_x\, n_x$$

Weak disorder: $0 < \eta \ll 1$.

C^*-algebra of bounded operators on $\mathfrak{F}$:

$$\mathfrak{A} := \overline{\{\text{bounded operators on } \mathfrak{F}\}}^{\|\cdot\|_{op}}$$

Consider a normalized, translation-invariant, deterministic state

$$\rho_0 : \mathfrak{A} \longrightarrow \mathbb{C}$$

preserving particle number, $\rho_0([A, N]) = 0 \; \forall A \in \mathfrak{A}$ ($N = \sum n_x$).

Define the time-evolved state

$$\rho_t(A) := \rho_0(e^{itH_\omega} A \, e^{-itH_\omega}) \, ,$$

and study in particular

$$\mathbb{E}[\rho_t(n_p)]$$

(expected number of electrons with momentum p at time t)

1.1. Thermodynamic and Kinetic Scaling Limit

Thm [C-Sasaki, JSP 08]

Assume ρ_0 number conserving + translation invariant.

For any $T > 0$ and all test functions f, g,

$$\Omega_T^{(2)}(\, f; g\,) \;:=\; \lim_{\eta\to 0} \lim_{L\to\infty} \mathbb{E}[\rho_{T/\eta^2}(\, a^+(f)\, a(g)\,)]$$

(macroscopic 2-point correlation) exists and is translation invariant.

Here,

$$a^+(f) \;:=\; \frac{1}{L^d} \sum_{\Lambda_L^*} f(p)\, a_p^+ \;\equiv\; \int dp\, f(p)\, a_p^+ \;=\; (a(f))^*$$

It defines inner product of f, g

$$\Omega_T^{(2)}(f;g) \;=\; \int_{\mathbb{T}^d} dp\, F_T(p)\, \overline{f(p)}\, g(p)$$

where $F_T(p)$ satisfies the linear Boltzmann equation

$$\partial_T F_T(p) \;=\; 2\,\pi \int_{\mathbb{T}^d} dp'\, \delta(\, E(p') - E(p)\,)\,(\, F_T(p') - F_T(p)\,)$$

with initial condition

$$F_0(p) \;=\; \lim_{L\to\infty}\, \rho_0(\, n_p\,)$$

(occupation density of momentum p) □

Outline of the Proof

Heisenberg evolution of the creation- and annihilation operators:

$$a(f,t) \;:=\; e^{itH_\omega} a(f) e^{-itH_\omega} \;=\; a(f_t)\,,$$

where f is a test function, and $a(f) \;=\; \int dp\, f(p)\, a_p$.

Expression $a(f_t)$ because H_ω is bilinear in a^+, a.

In particular, $a(f_0) = a(f)$, and

$$\begin{aligned} i\partial_t a(f_t) \;&=\; [\, H_\omega \,,\, a(f_t)\,] \\ &=\; a(\,\Delta f_t\,) + a(\,\eta\,\omega_x\, f_t\,)\,, \end{aligned}$$

Δ is the nearest neighbor Laplacian on Λ_L.

Thus, f_t solves the random Schrödinger equation (Anderson model)

$$i\partial_t f_t \;=\; \Delta f_t \,+\, \eta\, \omega_x f_t$$

$$f_0 \;=\; f\,.$$

Strategy: Determine dynamics of test fcts, f_t, g_t. Subsequently,

$$\begin{aligned}
\rho_t(\, a^+(f)\, a(g)\,) \quad &= \quad \rho_0(\, a^+(f_t)\, a(g_t)\,) \\
&= \quad \int dp\, dq\; \underbrace{\rho_0(\, a_p^+\; a_q)}_{\delta(p-q)\; J(p)}\; \overline{f_t(p)}\, g_t(q) \\
&= \quad \int dp\, J(p)\, \overline{f_t(p)}\, g_t(p)
\end{aligned}$$

By the Pauli principle (momentum p occupied by ≤ 1 fermion),

$$0 \leq J(p) = \rho_0(n_p) \leq 1 .$$

Example of free Gibbs state: $J(p) = \frac{1}{1+e^{\beta(E(p)-\mu)}}$, Fermi-Dirac distribution.

Analysis similar to Boltzmann limit for weakly disordered Anderson model !

Related works

(dynamics of Anderson model, excluding localization)

- Spohn [77]
- Erdös-Yau [00], Erdös [02], Erdös-Salmhofer-Yau [05]
- Lukkarinen [04], Lukkarinen-Spohn [05]
- Poupaud-Vasseur [03]
- Bourgain [02, 03]
- Shubin-Schlag-Wolff [02]
- Rodnianski-Schlag [03], Denisov [04], G. Perelman [04]
- C [05, 05, 06]

Duhamel series and Feynman graphs

Pick $N \in \mathbb{N}$, to be optimized later.

Write solutions f_t, g_t of random Schrodinger eq, with test functions as initial data, as truncated Duhamel series,

$$f_t \, = \, f_t^{(\leq N)} \, + \, f_t^{(>N)} \, ,$$

with remainder term $f_t^{(>N)}$, and

$$f_t^{(\leq N)} \, := \, \sum_{n=0}^{N} f_t^{(n)} \, .$$

Duhamel term of order $O(\eta^n)$ is given by

$$\widehat{f}_t^{(n)}(p) \quad := \quad \eta^n \, e^{\epsilon t} \int d\alpha \, e^{it\alpha} \int dk_0 \cdots dk_n \, \delta(p - k_0)$$
$$\Big[\prod_{j=0}^{n} \frac{1}{E(k_j) - \alpha - i\epsilon} \Big] \Big[\prod_{j=1}^{n} \widehat{V}_\omega(k_j - k_{j-1}) \Big] \, \widehat{f}(k_n) \, .$$

in resolvent form, and in frequency space representation.

From contour deformation, and to keep $e^{\epsilon t}$ bounded $\forall \, t$,

$$\epsilon \, = \, \frac{1}{t}$$

Induces expansion of pair correlation,

$$\rho_t(\, a^+(f)\, a(g)\,) \;=\; \rho_0(\, a^+(f_t)\, a(g_t)\,) \;=\; \sum_{n,\widetilde{n}\in\mathcal{I}_N} \rho_0(\, a^+(f_t^{(n)})\, a(g_t^{(\widetilde{n})})\,)$$

for $\mathcal{I}_N := \{1, \ldots, N, > N\}\,.$

Thus, for $n, \widetilde{n} \leq N$, $\bar{n} := \frac{n+\widetilde{n}}{2} \in \mathbb{N}$ (and $\widehat{V}_\omega(u)^* = \widehat{V}_\omega(-u)$),

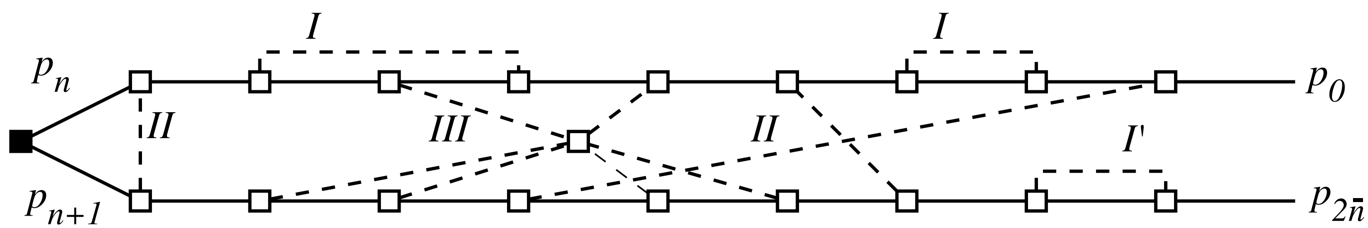
$$\mathbb{E}[\rho_0(\, a^+(f_t^{(n)})\, a(g_t^{(\widetilde{n})})\,)] \;=\; \eta^{2\bar{n}}\, e^{2\epsilon t} \int d\alpha\, d\widetilde{\alpha}\, e^{it(\alpha-\widetilde{\alpha})}$$

$$\int dp_0 \cdots dp_{2\bar{n}+1}\, \overline{f(p_0)}\, g(p_{2\bar{n}+1})\, J(p_n)\, \delta(p_n - p_{n+1})$$

$$\prod_{j=0}^{n} \frac{1}{E(p_j) - \alpha - i\epsilon} \prod_{\ell=n+1}^{2\bar{n}+1} \frac{1}{E(p_\ell) - \widetilde{\alpha} + i\epsilon}$$

$$\mathbb{E}\Big[\, \prod_{j=1}^{n} \widehat{V}_\omega(p_j - p_{j-1}) \prod_{j=n+2}^{2\bar{n}+1} \widehat{V}_\omega(p_j - p_{j-1}) \Big]$$

similar as for Anderson model ! $\Rightarrow$ Feynman graph expansion.



Proof strategy:

Complicated, high-dimensional singular integrals (resolvents !!).

Classification of Feynman graphs ([EY,ESY,C]):

Crossing and nesting diagrams yield small error terms.

Decorated ladder diagrams are dominant.

Sum of Feynman amplitudes of decorated ladder diagrams yields solution of linear Boltzmann equation. □

1.1.1. Discussion of Result

Consider Gibbs state for a free fermion gas,

$$\rho_0(A) = \frac{1}{Z_{\beta,\mu}} \mathrm{Tr}(e^{-\beta(T-\mu N)} A) \quad , \quad Z_{\beta,\mu} := \mathrm{Tr}(e^{-\beta(T-\mu N)})$$

at inverse temperature β, and with chemical potential μ.

Main observation:

Momentum distribution (Fermi-Dirac) in free Gibbs state

$$F_0(p) = \lim_{L\to\infty} \rho_0(n_p) = \frac{1}{1 + e^{\beta(E(p)-\mu)}}$$

is a *stationary solution* of the Boltzmann eq, $\forall\, 0 < \beta \leq \infty$.

Also true in zero temperature limit $\beta \to \infty$ (in the weak sense)

$$\frac{1}{1 + e^{\beta(E(p) - \mu)}} \to \chi[E(p) < \mu] \,.$$

Nontrivial provided that $\mu > 0$.

Question:

Under time evolution generated by H_ω, does the momentum distribution of the free Gibbs state drift towards a new equilibrium with a smaller occupation probability of high momenta ?

Diffusive drift due to localizing effect of random potential ?

Answer:

Not in kinetic time scale; momentum distribution persists.

Method suggests persistence into diffusive time scale ([ESY]).

1.2. Persistence of Quasifreeness

A state ρ_0 is quasifree (determinantal) if

$$\rho_0(\, a^+(f_1) \cdots a^+(f_r) a(g_1) \cdots a(g_s)\,)$$

$$= \delta_{r,s} \det \big[\, \rho_0(\, a^+(f_i) a(g_j)\,)\,\big]_{1 \le i,j \le r} \, .$$

Easy to show that $\rho_t(A)$ is quasifree *with probability 1.*

But: Almost sure quasifreeness $\not\Rightarrow \mathbb{E}[\rho_t(\,\cdot\,)]$ is quasifree.
(quasifreeness is a *nonlinear* condition on determinants !)

In fact, $\mathbb{E}[\rho_t(\,\cdot\,)]$ is *not* quasifree for any $\eta > 0$.

However, it possesses a quasifree kinetic scaling limit:

Thm [C-Sasaki, JSP 08]

Assume ρ_0 number conserving + translation invariant + quasifree.

Then, for all test functions f_j, g_ℓ, and any $T > 0$, $2r$-correlation fct

$$\Omega_T^{(2r)}(\, f_1, \dots, f_r \,;\, g_1, \dots, g_r \,)$$

$$:= \lim_{\eta \to 0} \lim_{L \to \infty} \mathbb{E}[\rho_{T/\eta^2}(\, a^+(f_1) \cdots a^+(f_r)\, a(g_1) \cdots a(g_r)\,)]$$

$$= \det \big[\, \Omega_T^{(2)}(\, f_i \,;\, g_j \,) \,\big]_{1 \le i,j \le r} \,,$$

with the macroscopic 2-point correlation as before,

$$\Omega_T^{(2)}(\, f_i \,;\, g_j \,) \;=\; \int dp\, F_T(p)\, \overline{f_i(p)}\, g_j(p) \,,$$

and $F_T(p)$ solves the previous linear Boltzmann equation. $\square$

Proof. Result

$$\lim_{\eta\to 0}\lim_{L\to\infty}\Big|\mathbb{E}[\rho_{T/\eta^2}(\,a^+(f_1)\cdots a^+(f_r)\,a(g_1)\cdots a(g_r)\,)]$$

$$-\det\big[\,\Omega_T^{(2)}(\,f_i\,;\,g_j\,)\,\big]_{1\le i,j\le r}\Big| \;=\; 0$$

Proof similar to:

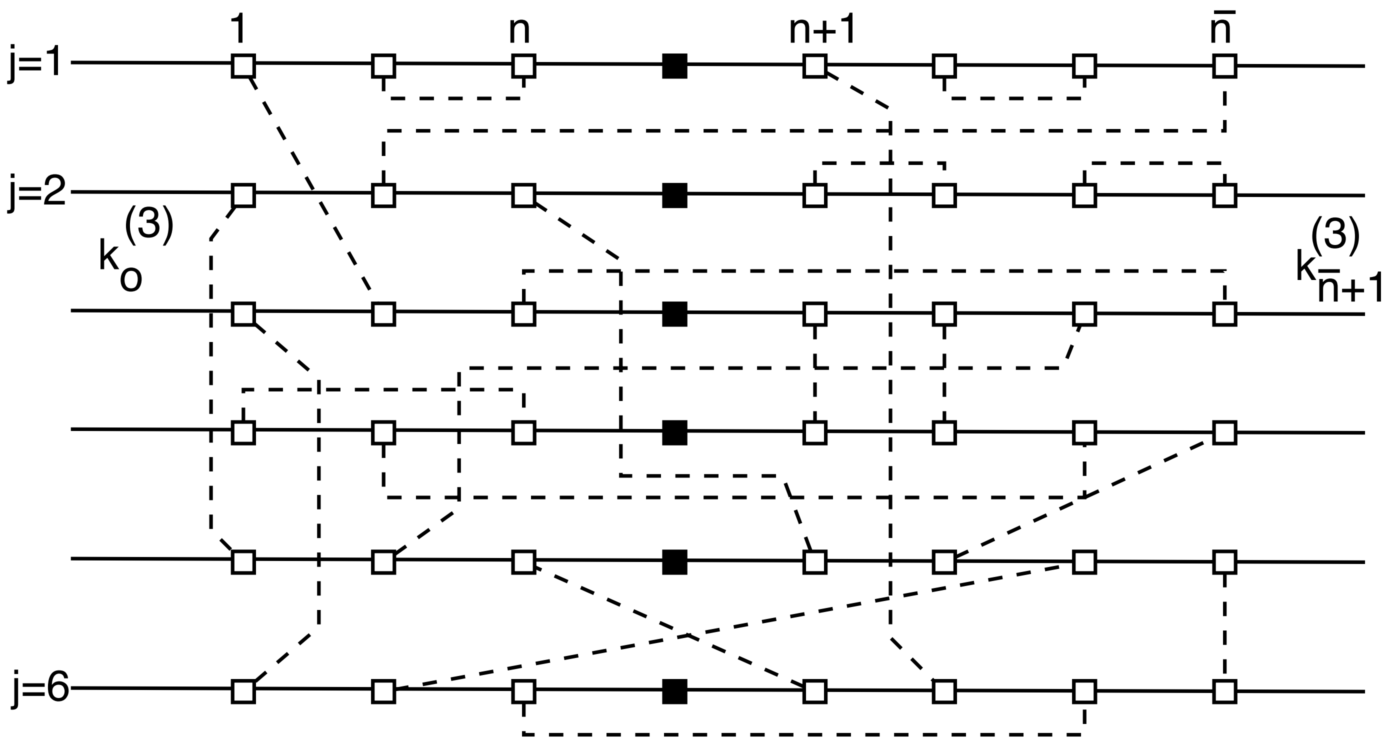
Thm [C, CMP '06] Globally in T, convergence in higher mean,

$$\lim_{\eta\to 0}\mathbb{E}\Big[\Big|\Big\langle\, W_T^{(\eta^2)}\,,\,J\,\Big\rangle-\Big\langle\,F_T\,,\,J\,\Big\rangle\Big|^r\,\Big] \;=\; 0$$

for any $1\le r<\infty$ for weakly disordered Anderson model.

Here, $W_T^{(\eta^2)}$ is the macroscopic rescaled Wigner transform, $F_T(X,V)$ solves a linear Boltzmann eq, and $J(X,V)$ is a test fct. $\square$

Among all Feynman graphs, only class of disconnected graphs is dominant and $O(1)$.



2. Interacting (mean field) Fermi gas in random medium

Joint work with I. Rodnianski.

Consider the time-dependent Hamiltonian

$$H(t) = T + \eta V_\omega + \lambda W(t)$$

where the fermion-fermion interaction is modeled by

$$W(t) = \sum_{x,y} v(x-y) \{ \mathbb{E}[\rho_t(a_x^+ a_x)]\, a_y^+ a_y - \mathbb{E}[\rho_t(a_y^+ a_x)]\, a_x^+ a_y \}$$

$\approx$ direct and exchange term (similar to Hartree-Fock approx.).

Dynamics of two-point correlation

$$
\begin{aligned}
& i\partial_t \, \rho_t(\, a_p^+ a_q \,) \\
= \quad & (\, E(p) - E(q) \,) \, \rho_t(\, a_p^+ a_q \,) \\
& + \lambda \int du \, \mathbb{E}[\, \rho_t(\, \frac{1}{L^d} a_u^+ a_u \,) \,] \, (\, \widehat{v}(u-p) \, \rho_t(\, a_u^+ a_q \,) \\
& \qquad\qquad\qquad\qquad - \widehat{v}(q-u) \, \rho_t(\, a_p^+ a_u \,) \,) \\
& + \eta \int du \, \widehat{\omega}(u-p) \rho_t(\, a_u^+ a_q \,) \; - \; \widehat{\omega}(q-u) \rho_t(\, a_p^+ a_u \,)
\end{aligned}
$$

Key observations:

- *Not* translation invariant for generic realization of V_ω.
  But translation invariant on $\mathbb{E}$-average !
- So far, equation does not close. But taking $\mathbb{E}$, it closes !

Translation invariant average

$$\mathbb{E}[\,\rho_t(\,a^+(f)\,a(g)\,)\,] \;=\; \int dp\,\overline{f(p)}\,g(p)\,\mu_t(p)$$

where

$$\mu_t(p) = \mathbb{E}[\rho_t(\,n_p\,)]\,,$$

momentum distribution function, averaged over random potential.

Dynamics of $\mu_t(p)$?

The *average*

$$\mathbb{E}[\rho_t(\,\cdot\,)] \,:\, \mathfrak{A} \to \mathbb{C}$$

solves

$$\begin{aligned} i\partial_t \, \mathbb{E}[\, \rho_t(\, A\,)\,] &= \mathbb{E}[\, \rho_t(\,[\, H(t)\,,\, A\,]\,)\,] \\ \mathbb{E}[\rho_0] &= \rho_0\,. \end{aligned}$$

May set $A = n_p$ by translation invariance.

Note that the Hamiltonian

$$H(t) \,=\, T \,+\, \eta\, V_\omega \,+\, \lambda\, W(t)$$

also depends on the unknown $\mu_t(p) = \mathbb{E}[\rho_t(\, n_p\,)]$.

$\Rightarrow$ Self-consistent nonlinear evolution equation for $\mu_t(p)$!

Again, use η (randomness) as Duhamel expansion parameter.

Now, the "free evolution" ($\eta = 0$ but $\lambda \neq 0$) is nonlinear !

Some key questions:

Dynamics at long time scales ?

Dependence of Boltzmann limit on ratio between λ and η ?

Effects of nonlinearity ?

Persistence of Fermi-Dirac distribution ?

The regime $\lambda \le C\eta^2$

The interaction between electrons and the effect of the random potential per time unit is comparable if $\lambda = C\eta^2$.

Thm [C-Rodnianski]

In the scaling limit determined by

$$t = \frac{T}{\eta^2} \quad , \quad \eta \to 0 \quad , \quad \lambda \le O(\eta^2) \, ,$$

the weak limit $\mathbb{E}[\rho_{T/\eta^2}(\,\cdot\,)] \to F$ *holds where*

$$\partial_T F_T(p) \, = \, 2\pi \int du \, \delta(\, E(u) - E(p)\,)\,(\, F_T(u) - F_T(p)\,)$$

with $F_0(p) = \mu_0(p)$.

The Hartree-Fock interactions *cancel*, due to translation invariance !

Proof:

Instead of free evolution $e^{i(t-s)E(p)}$, use

$$U_{s,t}(p) \;:=\; e^{i\int_s^t ds' \,(\, E(p) - \lambda \widehat{v} * \mu_{s'}(p)\,)}$$

and carry out Feynman graph expansion in powers of η.

<u>*Main difficulties:*</u>

- Free evolution operator depends on unknown $\mu_t(p)$, and satisfies *nonlinear* evolution equation

  $\Rightarrow$ Resolvent calculus *unvailable* !

  $\Rightarrow$ Entire analysis is based on *stationary phase estimates*.

- Recombination of decorated ladders much more complicated due to *nonlinear* "free" evolution.

  $\Rightarrow$ Phase cancellations and stationary phase. $\square$

Lemma Let $\kappa_s := \widehat{v} * \mu_s$. Then, uniformly in $\tau \geq 0$,

$$\Big| \int_{\mathbb{R}^+} ds\, e^{-is(E(u)-\alpha-i\epsilon)}\, e^{-i\lambda \int_\tau^{\tau+s} \kappa_{s'}(u)} ds' \Big| < \big(1 + \frac{\lambda}{\epsilon}\big) \frac{C}{|E(u)-\alpha|+\epsilon},$$

where $E(u)$ is the symbol of the nearest neighbor Laplacian on $\mathbb{Z}^3$.

Sketch of proof. We define

$$\overline{\kappa}_{t,t+s}(u) \; := \; \frac{1}{s} \int_t^{t+s} ds'\, \kappa_{s'}(u)\,.$$

Pauli principle $\Rightarrow |\overline{\kappa}_{t,t+s}(u)| < C_0$, uniformly in t and $s \geq 0$.

The integral on the left hand side of (1) can be written as

$$(*) \; := \int_{\mathbb{R}^+} ds\, e^{-is(E(u)-\alpha+\lambda\overline{\kappa}_{t,t+s}(u))} e^{-\epsilon s}\,.$$

To estimate it, we split $\mathbb{R}_+$ into disjoint intervals

$$I_j \; := \; [\, j\zeta\,,\, (j+1)\zeta\,) \quad , \quad j \in \mathbb{N}_0$$

of length

$$\zeta \ := \ \frac{\pi}{|E(u)-\alpha|}\,.$$

We find

$$(*) \quad = \quad \sum_{j\in 2\mathbb{N}_0} \int_{I_j} ds \left(e^{-is(E(u)-\alpha+\lambda\overline{\kappa}_{t,t+s}(u))} e^{-\epsilon s} \right.$$
$$\left. + \, e^{-i(s+\zeta)(E(u)-\alpha+\lambda\overline{\kappa}_{t,t+s+\zeta}(u))} e^{-\epsilon(s+\zeta)} \right),$$

where the second term in the bracket accounts for the integrals over I_j with j odd.

Evidently, $e^{-i\zeta(E(u)-\alpha)} \ = \ e^{\mp i\pi} \ = \ -1.$

Therefore, for j fixed,

$$\int_{I_j} ds \Big(e^{-is(E(u)-\alpha+\lambda\overline{\kappa}_{t,t+s}(u))} e^{-\epsilon s}$$

$$+ e^{-i(s+\zeta)(E(u)-\alpha+\lambda\overline{\kappa}_{t,t+s+\zeta}(u))} e^{-\epsilon(s+\zeta)} \Big)$$

$$\begin{aligned} = \quad & \int_{I_j} ds\, e^{-is(E(u)-\alpha+\lambda\overline{\kappa}_{t,t+s}(u))} \big(e^{-\epsilon s} - e^{-\epsilon(s+\zeta)} \big) \\ + \quad & \int_{I_j} ds\, e^{-is(E(u)-\alpha)} e^{-\epsilon(s+\zeta)} \big(e^{-i\lambda s\overline{\kappa}_{t,t+s}(u)} \;-\; e^{-i\lambda(s+\zeta)\overline{\kappa}_{t,t+s}(u)} \big) \\ + \quad & \int_{I_j} ds\, e^{-is(E(u)-\alpha)} e^{-\epsilon(s+\zeta)} \big(e^{-i\lambda(s+\zeta)\overline{\kappa}_{t,t+s}(u)} \;-\; e^{-i\lambda(s+\zeta)\overline{\kappa}_{t,t+s+\zeta}(u)} \big) \\ =: \quad & (*)_1 \,+\, (*)_2 \,+\, (*)_3 \,. \end{aligned}$$

Clearly,

$$\sum_{j\in 2\mathbb{N}_0} |(*)_1| \,<\, \int_{\mathbb{R}_+} ds\, e^{-\epsilon s}\, \epsilon\, \zeta \;=\; \frac{\pi}{|E(u)-\alpha|}\,,$$

and

$$\sum_{j\in 2\mathbb{N}_0} |(*)_2| \, < \, \int_{\mathbb{R}_+} ds \, e^{-\epsilon s} \, \lambda \, \zeta \, = \, \frac{\lambda}{\epsilon} \, \frac{\pi}{|E(u) - \alpha|} \, .$$

For $(*)_3$, we observe that for $s_1 < s_2$,

$$\begin{aligned} \overline{\kappa}_{t,t+s_2}(u) - \overline{\kappa}_{t,t+s_1}(u) \quad &= \quad (\frac{1}{s_2} - \frac{1}{s_1}) \int_t^{t+s_2} ds' \, \kappa_{s'}(u) \\ &+ \quad \frac{1}{s_1} \Big(\int_t^{t+s_2} - \int_t^{t+s_1} \Big) \, ds' \, \kappa_{s'}(u) \, . \end{aligned}$$

Since $|\kappa_{s'}(u)| < C_0$ uniformly in s', we immediately obtain

$$|\overline{\kappa}_{t,t+s_2}(u) - \overline{\kappa}_{t,t+s_1}(u)| \, < \, C \, \frac{s_2 - s_1}{s_1} \, ,$$

so that in particular,

$$|\overline{\kappa}_{t,t+\zeta+s}(u) - \overline{\kappa}_{t,t+s}(u)| \, < \, C \, \frac{\zeta}{s} \, .$$

Thus, we conclude that

$$\sum_{j\in 2\mathbb{N}_0} (*) \;\; \leq \;\; C\,\zeta\,\lambda \int_{\mathbb{R}_+} ds\, \frac{(s+\zeta)}{s}\, e^{-\epsilon(s+\zeta)}$$

$$\leq \;\; C\, \frac{\pi}{|E(u)-\alpha|}\, \frac{\lambda}{\epsilon}\,.$$

This proves that for $|E(u)-\alpha|>0$ and $\lambda = O(\epsilon)$,

$$|(*)| < \frac{C}{|E(u)-\alpha|}\,.$$

If $|E(u)-\alpha| \leq \epsilon$, then the trivial bound

$$|(*)| < \int_{\mathbb{R}_+} ds\, e^{-\epsilon s} < \frac{C}{\epsilon}$$

is better, which ignores phase cancellations.

In conclusion,

$$\Big| \int_{\mathbb{R}+} ds\, e^{-is(E(u)-\alpha-i\epsilon)}\, e^{-i\lambda \int_\tau^{\tau+s} \kappa_{s'}(u) ds'} \Big| < \Big(1+\frac{\lambda}{\epsilon}\Big) \frac{C}{|E(u)-\alpha|+\epsilon}\,,$$

as claimed. □

Use this estimate to adapt some resolvent estimates for the linear case. This allows to control error terms (non-ladder diagrams).

To control dominant terms (decorated ladder), can't lose the information about the phase (can't afford absolute values)

$\Rightarrow$ explicit stationary phase analysis.

The regime $\eta = o(\sqrt{\lambda})$

In this regime, the limiting distribution is *stationary*.

Thm [C-Rodnianski]

In the scaling limit determined by

$$t = \frac{T}{\lambda} \quad , \quad \lambda \to 0 \quad , \quad \eta = o(\sqrt{\lambda}) \, ,$$

the weak limit $\mathbb{E}[\rho_{T/\lambda}(\,\cdot\,)] \to F_T$ *holds, where*

$$\partial_T \, F_T(p) \, = \, 0$$

with initial condition $F_0(p) = \mu_0(p)$.

The regime $t = T/\eta^2$ and $\lambda = O_\eta(1)$

This regime is very difficult to control.

Partial result: Characterization of stationary solutions.

Fixed point equation: Let

$$\mu_t(p) \ := \ \frac{1}{L^d}\, \mathbb{E}[\, \rho_t(\, a_p^+ a_p \,)\,]\,.$$

Expand right hand side of

$$\begin{aligned} \int dp\, \overline{f(p)}\, g(p)\, \mu_t(p) \quad &= \quad \mathbb{E}[\, \rho_0(\mathcal{U}_t^*\, a^+(f) a(g)\, \mathcal{U}_t\,)\,] \\ &= \quad \mathcal{G}[\, \mu_\bullet\,;\, \eta\,;\, \lambda\,;\, t\,;\, f\,;\, g\,] \end{aligned}$$

into truncated Duhamel series; *fixed point equation* for μ_t.

Thm [C-Rodnianski]

Assume there exists a stationary fixed point

$$F(p) \;=\; F_T(p) \;\equiv\; \mu_0(p)$$

in the kinetic scaling limit determined by

$$t \;=\; \frac{T}{\eta^2} \quad , \quad \eta \to 0 \quad , \quad \lambda \le O(1)\,.$$

Then, it satisfies

$$F(p) \;=\; 2\pi \int du\, \delta(\, \widetilde{E}_\lambda(u) - \widetilde{E}_\lambda(p)\,)\, F(u)$$

where $\widetilde{E}_\lambda(p) \;=\; E(p) - \lambda(\,\widehat{v} * F\,)(p)$.

Energy renormalization !

Outlook

Dynamical equations for scaling $t = T/\eta^2$ and $\lambda = O(1)$.

$\Rightarrow$ *Very* difficult problem.

Spatially inhomogenous initial data.

More detailed study of diffusive regime.