A rigorous approach to the Chern-Simons path integral

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Motivation: Why is the theory interesting?

1) It is beautiful: surprising relations between many different areas of mathematics/physics like

- Algebra
- low-dimensional Topology
- Differential Geometry
- Functional Analysis and Stochastic Analysis
- Quantum field theory (in particular, Conformal field theory, Quantum Gravity, String theory)

2) It is deep: Fields Medals for Jones, Witten, Kontsevich

3) It is useful: Applications in Knot Theory and Quantum Gravity, ...

List of approaches

Original heuristic approach

0. Chern-Simons path integrals approach (Witten)

Rigorous perturbative approaches

1. Configuration space integrals
2. Kontsevich Integral

Rigorous non-perturbative approaches

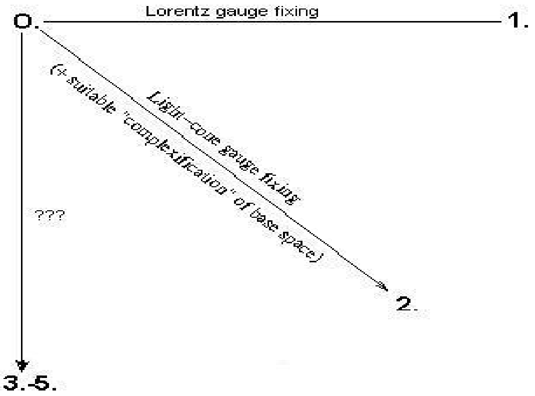
3a. Quantum groups + Surgery (Reshetikhin/Tureav)

3b. Quantum groups + Shadow links (Turaev)

3c. Lattice gauge theories based on Quantum groups

4. Skein Modules
5. "Sheaf of Vacua" Construction

Some Relations between the approaches



Important open problems

(P1) Chern-Simons path integral $\overset{??}{\longleftrightarrow}$ rigorous non-perturbative approaches 3a, 3b, 3c, 4, 5.

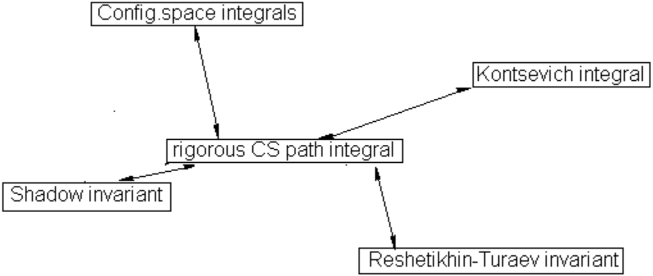
"How do quantum groups arise from path integrals?"

(P2) Rigorous definition of original Chern-Simons path integral expressions?

Alternatively:

(P2') Rigorous definition of Chern-Simons path integral expressions after suitable gauge fixing?

The longterm goal



(all "arrows" being rigorous)

2 Witten's Chern-Simons path integral approach

Fix

- M: oriented connected 3-manifold (usually compact)
- G: simply-connected Lie subgroup of $U(N)$ ($N \in \mathbb{N}$ fixed)
- $k \in \mathbb{R}\backslash\{0\}$ (usually $k \in \mathbb{N}$)

Space of gauge fields:

$$\mathcal{A} = \{A \mid A \ \mathfrak{g}\text{-valued 1-form on } M\} \quad (\mathfrak{g} \subset u(N)\text{: Lie algebra of } G)$$

Action functional:

$$S_{CS} : \mathcal{A} \ni A \mapsto \frac{k}{4\pi} \int_M \operatorname{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) \in \mathbb{R}$$

where $\operatorname{Tr} := c \operatorname{Tr}_{\mathrm{Mat}(N,\mathbb{C})}$ for suitable normalisation constant $c \in \mathbb{R}$

Observation 1

S_{CS} is invariant under (orientation-preserving) diffeomorphisms

Fix

- "link" $L = (l_1, l_2, \ldots, l_n)$, $n \in \mathbb{N}$, in M
- n-tuple $(\rho_1, \rho_2, \ldots, \rho_n)$ of finite-dim. representations of G

"Definition"

$$Z(M, L) := \int \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(A)) \exp(iS_{CS}(A)) DA$$

where DA is the "Lebesgue measure" on $\mathcal{A}$ and

$$\mathrm{Hol}_{l_i}(A) := \lim_{n \to \infty} \prod_{k=1}^{n} \exp\left(\tfrac{1}{n} A\left(l_i'\left(\tfrac{k}{n}\right)\right)\right) \quad (\text{"holonomy of } A \text{ around } l_i\text{"})$$

Observation 2

S_{CS} invariant under (orientation-preserving) diffeomorphisms $\Rightarrow$ $Z(M, L)$ only depends on diffeomorphism class of M and isotopy class of L.

3 Turaev's "shadow world" approach

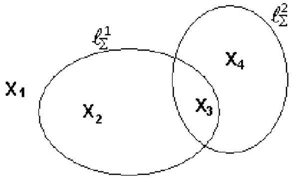
Special case: $M = \Sigma \times S^1$

Fix (framed) link $L = (l^1, l^2, \ldots, l^n)$

Loop projections onto S^1 and Σ:

$$l^1_{S^1}, l^2_{S^1}, ..., l^n_{S^1} \qquad \text{and} \qquad l^1_\Sigma, l^2_\Sigma, ..., l^n_\Sigma$$

$D(L)$: graph in Σ generated by l^1_Σ, l^2_Σ, ..., l^n_Σ



$X_1, X_2, \ldots, X_m$: "faces" in $D(L)$

Gleams

Each X_t is equipped in a canonical way with a "gleam" $\mathrm{gl}_t \in \mathbb{Z}$
Gleams $(\mathrm{gl}_t)_t$ contain

- Information about crossings in $D(L)$,
- Information about winding numbers $\mathrm{wind}(l^j_{S^1})$

"shadow of L"

$sh(L) := (D(L), (\mathrm{gl}_t)_t)$

Example 1: $D(L)$ has no crossing points

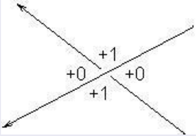
$$\mathrm{gl}_t = \sum_{\{j|\ l^j_\Sigma \text{ touches } X_t\}} \mathrm{wind}(l^j_{S^1}) \cdot \mathrm{sgn}(X_t; l^j_\Sigma)$$

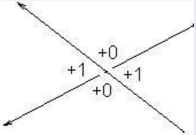
where

$$\mathrm{sgn}(X_t; l^j_\Sigma) = \begin{cases} 1 & \text{if } X_t \text{ is "inside" of } l^j_\Sigma \\ -1 & \text{if } X_t \text{ is "outside" of } l^j_\Sigma \end{cases}$$

Example 2: $D(L)$ has crossing points but $\mathrm{wind}(l^j_{S^1}) = 0$, $j \leq n$

Figure: Changes in the gleams at a given crossing point





Let $\mathfrak{g}$ and k be as in Sec. 2. Fix Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$.

Colors and Colorings

- "color": dominant weight of $\mathfrak{g}$ (w.r.t. $\mathfrak{t}$) which is "integrable at level" k.
- $\mathcal{C}$: set of colors
- "link coloring" : mapping $\gamma : \{l_1, l_2, \ldots, l_n\} \to \mathcal{C}$
- "area coloring": mapping $\text{col} : \{X_1, \ldots, X_m\} \to \mathcal{C}$.
- Col: set of area colorings

Fix link coloring $\gamma : \{l_1, l_2, \ldots, l_n\} \to \mathcal{C}$.

Example 3: $\mathfrak{g} = su(2)$, $\mathfrak{t}$ arbitrary

$\mathcal{C} \cong \{0, 1, 2, \ldots, k - 2\}$

"Fusion coefficients" $N^{\gamma}_{\alpha\beta} \in \mathbb{N}_0, \quad \alpha, \beta, \gamma \in \mathcal{C}$

$$N^{\gamma}_{\alpha\beta} = \sum_{\sigma \in W_k} (-1)^{\sigma} m_{\alpha}(\beta - \sigma(\gamma)),$$

where

- $m_{\alpha}(\beta)$: multiplicity of weight β in character of α.
- W_k: "quantum Weyl group" at level k

Remark

More frequently the following (equivalent) definition of $N^{\gamma}_{\alpha\beta}$ is used:

$$N^{\gamma}_{\alpha\beta} = \sum_{\delta} \frac{S_{\alpha\delta} S_{\beta\delta} S_{\gamma^* \delta}}{S_{\rho\delta}} \qquad \alpha, \beta, \gamma \in \mathcal{C},$$

where $(S_{\alpha\beta})_{\alpha\beta}$ is the S-matrix of the "associated" CFT and where ρ is the Weyl vector.

"Shadow invariant" $|\cdot|$ for $\mathfrak{g}$ and k

$$|L| = \sum_{\mathrm{col} \in Col} \Big(\prod_{i=1}^{n} N^{\mathrm{col}(Y_i^-)}_{\gamma(l_i)\,\mathrm{col}(Y_i^+)}\Big) \left(\prod_{t=1}^{m} (V_{\mathrm{col}(X_t)})^{\chi(X_t)} \exp(2\,\mathrm{gl}_t\, U_{\mathrm{col}(X_t)})\right)$$

$$\times \left(\prod_{p \in \mathrm{DP}(L)} \mathrm{symb}_q(\mathrm{col}, p)\right) \quad \text{where}$$

$\chi(X_t)$: Euler characteristic of X_t

$Y_i^{+/-}$: face touching l^i_Σ from "inside" / "outside"

$V_\lambda := \prod_{\alpha \in R_+} \frac{\sin \frac{\pi \langle \lambda+\rho, \alpha \rangle}{k}}{\sin \frac{\pi \langle \rho, \alpha \rangle}{k}}$ where R_+ positive roots and

$U_\lambda := \exp(\frac{\pi i}{k} \langle \lambda, \lambda + 2\rho \rangle)$ $\langle \cdot, \cdot \rangle$ suitably normalized Killing form

$\mathrm{symb}_q(\mathrm{col}, p)$: associated q-6j-symbol for $q := \exp(\frac{2\pi i}{k})$

Example: $G = SU(2)$ (with $\mathcal{C} \cong \{0, 1, \ldots, k-2\}$)

- $U_\lambda = \frac{\pi i}{k} \lambda(\lambda + 1)$
- $V_\lambda = \frac{\sin((\lambda+1)\pi/k)}{\sin(\pi/k)}$
- $N^{\gamma}_{\alpha\beta} \in \{0, 1\} \quad \Rightarrow$

$$|L| = \sum_{\mathrm{col} \in Col'} \left(\prod_{t=1}^{m} (V_{\mathrm{col}(X_t)})^{\chi(X_t)} \exp(2\, \mathrm{gl}_t\, U_{\mathrm{col}(X_t)}) \right) \left(\prod_{p \in \mathrm{DP}(L)} \mathrm{symb}_q(\mathrm{col}, p) \right)$$

where Col' is a suitable subset of Col.

Special case: $D(L)$ has no crossing points (G is general)

$$|L| = \sum_{\mathrm{col} \in Col} \left(\prod_{i=1}^{n} N^{\mathrm{col}(Y_i^-)}_{\gamma(l_i)\,\mathrm{col}(Y_i^+)} \right) \left(\prod_{t=1}^{m} (V_{\mathrm{col}(X_t)})^{\chi(X_t)} \exp(2\, \mathrm{gl}_t\, U_{\mathrm{col}(X_t)}) \right)$$

4 From path integrals to the "shadow world"

Gauge group: $\mathcal{G} = C^\infty(M, G)$

$\mathcal{G}$ operates on $\mathcal{A}$ from the right by

$$A \cdot \Omega = \Omega^{-1} A \Omega + \Omega^{-1} d\Omega \qquad \text{for} \quad \Omega \in \mathcal{G}, A \in \mathcal{A}$$

Gauge Fixing: Choice of system $\mathcal{A}_{\text{gf}}$ of representatives of $\mathcal{A}/\mathcal{G}$

Example: "Axial gauge fixing" for $M = \mathbb{R}^3$

In this case each $A \in \mathcal{A}$ can be written as $A = \sum_{i=0}^{2} A_i dx_i$.

$$\mathcal{A}_{\text{gf}} = \{A \mid A_0 = 0\}$$

is "essentially" a gauge-fixing.

"Faddeev-Popov determinant"

If $\mathcal{A}_{\text{gf}}$ is "nice" enough there is a function $\triangle_{FadPop} : \mathcal{A}_{\text{gf}} \to \mathbb{R}$ such that (informally)

$$\int_{\mathcal{A}} \chi(A) DA = \int_{\mathcal{A}_{\text{gf}}} \chi(A) \triangle_{FadPop}(A) DA_{|\mathcal{A}_{\text{gf}}}$$

for every $\mathcal{G}$-invariant function $\chi : \mathcal{A} \to \mathbb{C}$

Example

For $M = \mathbb{R}^3$ and $\mathcal{A}_{\text{gf}} := \mathcal{A}^{\perp} := \{A \in \mathcal{A} \mid A_0 = 0\}$ we have

$$\triangle_{FadPop}(A) = const. \qquad \Rightarrow$$

$$\int_{\mathcal{A}} \chi(A) DA \sim \int_{\mathcal{A}^{\perp}} \chi(A^{\perp}) DA^{\perp}, \qquad \text{with } DA^{\perp} := DA_{|\mathcal{A}^{\perp}}$$

Example for usefulness of applying a gauge fixing

For $M = \mathbb{R}^3$ and $\mathcal{A}_{gf} := \mathcal{A}^\perp := \{A \in \mathcal{A} \mid A_0 = 0\}$ we have

$$
\begin{aligned}
Z(M, L) &= \int_{\mathcal{A}} \prod_i \mathrm{Tr}(\mathrm{Hol}_{l_i}(A)) \exp(iS_{CS}(A)) DA \\
&\sim \int_{\mathcal{A}^\perp} \prod_i \mathrm{Tr}(\mathrm{Hol}_{l_i}(A)) \exp(iS_{CS}(A)) DA^\perp \\
&\overset{(*)}{=} \int_{\mathcal{A}^\perp} \prod_i \mathrm{Tr}(\mathrm{Hol}_{l_i}(A^\perp)) \exp(i\tfrac{k}{4\pi} \int \mathrm{Tr}(dA^\perp \wedge A^\perp)) DA^\perp
\end{aligned}
$$

$(*)$ holds because $A^\perp \wedge A^\perp \wedge A^\perp = 0$ for $A^\perp \in \mathcal{A}^\perp$.

The last integral involves a **Gauss-type measure!**

Torus Gauge

$M = \mathbb{R}^3$

$A = \sum_{i=0}^{2} A_i dx_i = A^\perp + A_0 dx_0$ where $A^\perp := A_1 dx_1 + A_2 dx_2$

Note that $A^\perp \in \mathcal{A}^\perp := \{A \in \mathcal{A} \mid A(\frac{\partial}{\partial x_0}) = 0\}$

$\mathcal{A}_{\text{gf}} = \mathcal{A}^\perp = \{A \mid A_0 = 0\}$ **Axial gauge fixing**

$M = \Sigma \times S^1$

Each $A \in \mathcal{A}$ we have $A = A^\perp + A_0 dt$ with $A_0 \in C^\infty(\Sigma \times S^1, \mathfrak{g})$ and $A^\perp \in \mathcal{A}^\perp := \{A \in \mathcal{A} \mid A(\frac{\partial}{\partial t}) = 0\}$

dt and $\frac{\partial}{\partial t}$ obtained by lifting obvious 1-form/vector field on S^1 to $\Sigma \times S^1$

Problem: $\mathcal{A}_{\text{gf}} = \mathcal{A}^\perp$ is not a gauge fixing! We need larger space

1. Option

$$\mathcal{A}_{gf} = \mathcal{A}^{\perp} \oplus \{B dt \mid B \in C^{\infty}(\Sigma, \mathfrak{g})\} \cong \mathcal{A}^{\perp} \oplus C^{\infty}(\Sigma, \mathfrak{g})$$

2. Option: "torus gauge" (Blau/Thompson '93)

$\mathcal{A}_{gf} = \mathcal{A}^{\perp} \oplus \{B dt \mid B \in C^{\infty}(\Sigma, \mathfrak{t})\} \cong \mathcal{A}^{\perp} \oplus C^{\infty}(\Sigma, \mathfrak{t})$ where $\mathfrak{t}$ is Lie algebra of fixed maximal torus $T \subset G$

Example: $T = \left\{ \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \cong U(1)$ is a max. torus for $G = SU(2)$

Observation

$\triangle_{FadPop}(A^{\perp} + Bdt)$ only depends on B and contains sin(.)-functions $\longrightarrow$ notation $\triangle_{FP}(B) := \triangle_{FadPop}(A^{\perp} + Bdt)$

Example: $\triangle_{FadPop}(A^{\perp} + Bdt) = \sin^2(B/i)$ for $G = SU(2)$ and T as above (so $\mathfrak{t} \cong i\mathbb{R}$).

Torus Gauge applied to CS theory on $M = \Sigma \times S^1$

"Definition" of $\triangle_{FP} \Rightarrow$

$$Z(M, L) = \int_{\mathcal{A}} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(A)) \exp(iS_{CS}(A)) DA$$

$$\sim \int_{C^\infty(\Sigma,\mathfrak{t})} \int_{\mathcal{A}^\perp} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(A^\perp + Bdt)) \exp(iS_{CS}(A^\perp + Bdt))$$

$$\times \triangle_{FP}(B) DA^\perp DB \quad \text{where}$$

- $DA^\perp$: "Lebesgue measure" on $\mathcal{A}^\perp$
- DB: "Lebesgue measure" on $C^\infty(\Sigma, \mathfrak{t})$

1. important Observation

$S_{CS}(A^\perp + Bdt)$ quadratic in $A^\perp$ for fixed B

$\mathcal{A}_{\Sigma,V} :=$ Space of V-valued 1-forms on Σ for $V \in \{\mathfrak{g}, \mathfrak{t}, \mathfrak{t}^\perp\}$

Identification $\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}})$

$$\mathcal{A}^\perp_c := \{A^\perp \in \mathcal{A}^\perp \mid A^\perp \text{ constant and } \mathcal{A}_{\Sigma,\mathfrak{t}}\text{-valued}\}$$

$$\check{\mathcal{A}}^\perp := \{A^\perp \in \mathcal{A}^\perp \mid \int_{S^1} A^\perp(t)dt \in \mathcal{A}_{\Sigma,\mathfrak{t}^\perp}\}$$

Decomposition $\mathcal{A}^\perp = \check{\mathcal{A}}^\perp \oplus \mathcal{A}^\perp_c$

2. important Observation

$$S_{CS}(\check{A}^\perp + A^\perp_c + Bdt) = S_{CS}(\check{A}^\perp + Bdt) + \frac{k}{2\pi} \int_\Sigma \mathrm{Tr}(dA^\perp_c \cdot B)$$

Final heuristic integral formula

$$Z(M,L) \sim \int_{C^\infty(\Sigma,\mathfrak{t})} \int_{\mathcal{A}_c^\perp} \int_{\check{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(\check{A}^\perp + A_c^\perp + B dt)) d\check{\mu}_B^\perp(\check{A}^\perp)$$

$$\times \triangle_{FP}(B) Z(B) \exp(i \tfrac{k}{2\pi} \int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B)) DA_c^\perp DB$$

where

$$d\check{\mu}_B^\perp(\check{A}^\perp) := \frac{1}{Z(B)} \exp(i S_{CS}(\check{A}^\perp + B dt)) D\check{A}^\perp$$

with $Z(B) := \int \exp(i S_{CS}(\check{A}^\perp + B dt)) D\check{A}^\perp$.

3. important Observation

Both $d\check{\mu}_B^\perp$ and $\exp(i \frac{k}{2\pi} \int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B)) DA_c^\perp DB$ are of "Gauss-type"

Rigorous implementation

The "continuum approach"

Use "White noise analysis"-framework in similar way as in Albeverio/Sengupta '97 and certain regularization techniques:

1. "Framing": Choose diffeomorphism $\phi : \Sigma \times S^1 \to \Sigma \times S^1$ such that

$$\phi \sim \mathrm{id}_{\Sigma \times S^1}, \qquad \phi^*(\mathcal{A}^\perp) = \mathcal{A}^\perp$$

   Deform $\check{\mu}^\perp_B \longrightarrow \check{\mu}^\perp_{B,\phi}$
2. Rigorous implementation of $\int \cdots d\check{\mu}^\perp_{B,\phi}$ as a Hida distribution on suitable extension $\overline{\check{\mathcal{A}}^\perp}$ of $\check{\mathcal{A}}^\perp$ (fixed auxiliary Riemannian metric on Σ)
3. "Loop smearing"
4. Regularization of $\triangle_{FP}(B)Z(B)$ (+ index theorem or result on Ray-Singer torsion) $\longrightarrow$ Euler characteristics $\chi(X_t)$ appear

The "discretization approach"

Fix $m \in \mathbb{N}$ and fix triangulation of Σ, K being the underlying simplicial complex.

Let $C^p(K, V)$ denote the space of V-valued p-cochains for K (for $p \in \mathbb{N}_0$ and Abelian group V)

Discretization based on replacements

- $\mathcal{B} = C^\infty(\Sigma, \mathfrak{t}) = \Omega^0(K, \mathfrak{t}) \longrightarrow C^0(K, \mathfrak{t})$
- $\mathcal{A}_{\Sigma,\mathfrak{g}} = \Omega^1(K, \mathfrak{g}) \longrightarrow C^1(K, \mathfrak{g})$
- $\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_{\Sigma,\mathfrak{g}}) \longrightarrow Maps(\mathbb{Z}_m, C^1(K, \mathfrak{g}))$

Remark

Works if G is of the form $G = G_0 \times G_0$ ("Field doubling")

Apart from K we also have to use the dual K' of K (cf. D.H. Adams' results for Abelian CS theory, '96)

Some Properties of framed heuristic measure $\check{\mu}^{\perp}_{B,\phi}$

- oscillatory complex measure of "Gauss-type"
- normalized
- zero mean
- non-definite covariance operator

Toy model: complex "Gauss-type" measure μ on $\mathbb{R}^2$

$$\mu(x) = \tfrac{1}{2\pi} \exp(i \tfrac{1}{2} \langle x, Cx \rangle) dx \quad \text{where } C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Clearly, $\langle v, Cv \rangle = 0$ for $v = (1, 0)$

$\Rightarrow \quad \lim_{\epsilon \to 0} \int \langle x, v \rangle^n e^{-\epsilon |x|^2} d\mu(x) = 0 \qquad$ for all $n \in \mathbb{N}$

$\Rightarrow \quad \lim_{\epsilon \to 0} \int \Phi(\langle x, v \rangle) e^{-\epsilon |x|^2} d\mu(x) = \Phi(0)$ (for all "sufficiently nice" entire analytic functions $\Phi : \mathbb{R} \to \mathbb{R}$)

Evaluation of the path integral

Recall:

- We have restricted ourselves to the special case L has no crossings
- We derived the heuristic formula

$$\begin{aligned} &\mathrm{Z}(M, L) \\ &\sim \int_{C^\infty(\Sigma,\mathfrak{t})} \int_{\mathcal{A}_c^\perp} \int_{\check{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(\check{A}^\perp + A_c^\perp + B dt)) d\check{\mu}_{B,\phi}^\perp(\check{A}^\perp) \\ &\qquad \times \triangle_{FP}(B) Z(B) \exp(i \tfrac{k}{2\pi} \int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B)) DA_c^\perp DB \end{aligned}$$

- We are able to make rigorous sense of the r.h.s. of $\mathrm{Z}(M, L)$

For simplicity: heuristic treatment

1. Step: Perform $\int \cdots d\check{\mu}^{\perp}_{B,\phi}(\check{A}^{\perp})$

$$\int_{\check{\mathcal{A}}^{\perp}} \prod_j \mathrm{Tr}_{\rho_j}(\mathrm{Hol}_{l_j}(\check{A}^{\perp} + A_c^{\perp} + Bdt)) d\check{\mu}^{\perp}_{B,\phi}(\check{A}^{\perp})$$

$$= \prod_j \mathrm{Tr}_{\rho_j}(\mathrm{Hol}_{l_j}(0 + A_c^{\perp} + Bdt)) = \prod_j \mathrm{Tr}_{\rho_j}(\exp(\int_{l_j} A_c^{\perp} + \int_{l_j} Bdt)$$

$\longrightarrow$

$$Z(M, L) \sim \int_{C^{\infty}(\Sigma,\mathfrak{t})} \int_{\mathcal{A}_c^{\perp}} \prod_j \mathrm{Tr}_{\rho_j}(\exp(\int_{l_j} A_c^{\perp} + \int_{l_j} Bdt))$$

$$\times \triangle_{FP}(B) Z(B) \exp(i\tfrac{k}{2\pi} \int_{\Sigma} \mathrm{Tr}(dA_c^{\perp} \cdot B)) DA_c^{\perp} DB$$

2. Step: Perform $\int \cdots DA_c^\perp$

Observe

1. $\mathrm{Tr}_{\rho_j}(e^b) = \sum_\alpha m_{\rho_j}(\alpha) e^{i\alpha(b)}$ if $b \in \mathfrak{t}$
2. $\mathrm{Tr}(dA_c^\perp \cdot B) = \ll B, \star dA_c^\perp \gg$
3. $\alpha(\int_{l_\Sigma^j} A_c^\perp) = \alpha(\int_{R_\Sigma^j} dA_c^\perp) = \ll \alpha \cdot 1_{R_\Sigma^j}, \star dA_c^\perp \gg$

$\longrightarrow$

$$\int \prod_j \mathrm{Tr}_{\rho_j}(\exp(\int_{l_j} A_c^\perp + \int_{l_j} B dt) \exp(i \tfrac{k}{2\pi} \int_\Sigma \mathrm{Tr}(dA_c^\perp \cdot B)) DA_c^\perp$$

$$= \sum_{\alpha_1,\ldots,\alpha_n} (\prod_j m_{\rho_j}(\alpha_j)) \quad [\ldots] \quad \int \exp(i \ll \tfrac{k}{2\pi} B - \sum_j \alpha_j 1_{R_\Sigma^j}, \ \star dA_c^\perp \gg) DA_c^\perp$$

$$\delta(B - \tfrac{2\pi}{k} \sum_j \alpha_j 1_{R_\Sigma^j})$$

3. Step: Perform $\int \cdots DB$

$$\mathrm{Z}(M, L)$$

$$\sim \sum_{\alpha_1,\ldots,\alpha_n} \int (\prod_j m_{\rho_j}(\alpha_j))(\triangle_{FP}(B)Z(B))(\exp(\ldots))\delta(B - \tfrac{2\pi}{k} \sum_j \alpha_j 1_{R^j_\Sigma})DB$$

$$\overset{(*)}{=} \sum_{\{B=\frac{2\pi}{k}\sum_j 1_{R^j_\Sigma}\alpha_j\}} (\prod_j m_{\rho_j}(\alpha_j))(\triangle_{FP}(B)Z(B))(\exp(\ldots))$$

$$= \quad \ldots$$

$$= \sum_{\mathrm{col} \in Col} \Big(\prod_{j=1}^{n} N_{\gamma(l_j)\,\mathrm{col}(Y_j^+)}^{\mathrm{col}(Y_j^-)}\Big) \Big(\prod_{t=1}^{m} (V_{\mathrm{col}(X_t)})^{\chi(X_t)}\Big) \Big(\prod_{t=1}^{m} \exp(2\,\mathrm{gl}_t\, U_{\mathrm{col}(X_t)})\Big)$$

$$= |L|$$

(step $(*)$ is not quite the full story, cf. the "Appendix" below)

Open Questions

- Case where L does have crossing points:
  Do we obtain quantum 6j-symbols $\mathrm{symb}_q(\mathrm{col}, p)$?
  Note: For fixed complex Lie algebra $\mathfrak{g}$ the deformation

  $$\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}_q(\mathfrak{g})$$

  involves fixed Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$

  Probably: $\mathfrak{t}$ comes from maximal torus T!
- Generalization to manifolds with boundary?
- Discretization approach possible for original (= non-gauge fixed) path integral?

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Appendix

Torus gauge fixing revisited

Topological obstructions

$\to$ strictly speaking torus gauge is not a gauge

$\to$ we must allow $A_c^{\perp}$ to have a singularity in fixed point σ_0 of Σ

$\to$ 1-1-correspondence

$$\{\text{relevant singularities of } A_c^{\perp} \text{ in } \sigma_0\} \quad \longleftrightarrow \quad [\Sigma, G/T] \cong \mathbb{Z}^{\dim(T)}$$

$\to$ extra summation $\sum_{h \in [\Sigma, G/T]} \cdots$ (plus a term depending on the "winding number" of h) in some formulas

$\to$ this extra summation (combined with a suitable application of the Poisson summation formula) does indeed lead to the correct expressions at the end of the "3. Step" on page 31