SPDEs: Regularity of the probability law of the solution

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Classical and Random Dynamics in Mathematical Physics
March 31 - April 3, 2010
The University of Texas at Austin

Stochastic partial differential equations

$$Lu(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x)$$

- $t \geq 0, x \in \mathbb{R}^d$
- L is a second order differential operator
- To simplify, assume zero initial conditions
- $\{\dot{W}(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a zero mean Gaussian generalized process with covariance

$$E(\dot{W}(t,x)\dot{W}(s,y)) = \delta_0(s-t)f(x-y),$$

where $f \geq 0$ is the Fourier transform of a non-negative definite tempered measure (*spectral measure*) μ on $\mathbb{R}^d$, that is, for some $m \geq 1$

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^m} < \infty$$

The noise

- If $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$W(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t,x) \dot{W}(t,x) dx dt, \tag{1}$$

  defines a Gaussian family of random variables with covariance

$$\begin{aligned} E(W(\varphi)W(\psi)) &= \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x) f(x-y) \psi(t,y) dx dy dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(t)(\xi) \overline{\mathcal{F}\psi(t)(\xi)} \mu(d\xi) dt \\ &= \langle \varphi, \psi \rangle_{\mathcal{H}} \end{aligned}$$

- Let $\mathcal{H}$ be the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Then the stochastic integral (1) can be extended to $\mathcal{H}$

Cylindrical Wiener process

- Note that $\mathcal{H} = L^2(\mathbb{R}_+; \mathcal{H}_0)$, where $\mathcal{H}_0$ is the completion of $C_0^\infty(\mathbb{R}^d)$ with the inner product

$$
\begin{aligned}
\langle \varphi, \psi \rangle_{\mathcal{H}_0} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) f(x-y) \psi(y) dx dy \\
&= \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)} \mu(d\xi)
\end{aligned}
$$

- Set $W_t(h) = W(1_{[0,t]}h)$ for any $t \geq 0$ and $h \in \mathcal{H}_0$. Then, $\{W_t, t \geq 0\}$ is a cylindrical Wiener process in the Hilbert space $\mathcal{H}_0$

Examples:

- If $f(x) = \delta_0(x)$, then μ is the Lebesgue measure and W is a space-time white noise. In this case $\mathcal{H}_0 = L^2(\mathbb{R}^d)$
- Let $0 < \beta < d$ and $f(x) = |x|^{-\beta}$ (*Riesz kernel*). Then

$$\mu(d\xi) = c_{d,\beta} \frac{d\xi}{|\xi|^{d-\beta}}$$

Stochastic integrals

- Let $\mathcal{F}_t = \sigma\{W_s(h), h \in \mathcal{H}_0, 0 \le s \le t\}$. The predictable σ-field in $\Omega \times \mathbb{R}_+$ is generated by the sets $\{(s,t] \times A,\ 0 \le s < t, A \in \mathcal{F}_s\}$
- For any predictable process $g \in L^2(\Omega \times \mathbb{R}_+; \mathcal{H}_0)$ the stochastic integral $\int_0^\infty \int_{\mathbb{R}^d} g(t,x) W(dt,dx)$ is well defined and

$$E\left(\left|\int_0^\infty \int_{\mathbb{R}^d} g(t,x) W(dt,dx)\right|^2\right) = E\left(\int_0^\infty \|g(t,\cdot)\|_{\mathcal{H}_0}^2 \, dt\right)$$

Mild solutions

Definition

A (*mild or evolution*) solution to Equation (1) is a predictable stochastic process $\{u(t,x),(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ satisfying

$$\begin{aligned} u(t,x) &= \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))W(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)b(u(s,y))dyds, \end{aligned}$$

where G denotes the fundamental solution associated to $Lu = 0$

Existence and uniqueness of solutions

Theorem (Dalang, 1999)

Suppose that G_t is a non-negative measure such that:

- *For all $T > 0$, $G_T(\cdot)$ has rapid decrease, and* $\sup_{0\leq t\leq T} G_t(\mathbb{R}^d) \leq C_T < \infty$
- *For all $T > 0$*

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}G_t(\xi)|^2 \mu(d\xi)dt < \infty \qquad (2)$$

Suppose that b and σ are Lipschitz functions. Then, Equation (1) has a unique mild solution $u(t,x)$ which is continuous in L^2 and satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u(t,x)|^p) < \infty$$

for all $T > 0$ and $p \geq 1$

- Property (2) implies that $G \in L^2([0,T];\mathcal{H}_0)$ for any $T > 0$. This property, together with the positivity of G, implies that $\{G_{t-s}(x-y)\sigma(u(s,y)), 0 \le s \le t, y \in \mathbb{R}^d\}$ is a predictable square integrable process in $L^2(\Omega \times [0,t];\mathcal{H}_0)$
- This result was extended by Conus and Dalang, 2008, to the case where G is a distribution, which satisfies, for all $T > 0$,

$$\int_0^T \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G_t(\xi+\eta)|^2 \mu(d\xi) \right) dt < \infty$$

However, for $d \ge 4$, it is not known in general if the solution has moments of all orders

The wave equation on $\mathbb{R}^d$

$$\frac{\partial^2 G^{(d)}}{\partial t^2} - \Delta G^{(d)} = 0$$

We have

$$\begin{aligned} G_t^{(1)}(x) &= \frac{1}{2}\mathbf{1}_{\{|x|<t\}}, \\ G_t^{(2)}(x) &= C(t^2 - |x|^2)_+^{-1/2}, \\ G_t^{(3)}(x) &= \frac{1}{4\pi t}\sigma_t(dx), \end{aligned}$$

where σ_t is the surface measure on the three-dimensional sphere of radius t. For all $d \geq 1$,

$$\mathcal{F}G_t^{(d)}(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}$$

The heat equation on $\mathbb{R}^d$:

$$\frac{\partial G}{\partial t} - \frac{1}{2}\Delta G = 0$$

G is given by the Gaussian density

$$G_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$$

and

$$\mathcal{F} G_t(\xi) = \exp(-4\pi^2 t|\xi|^2)$$

- In both examples G satisfies condition (2) if and only if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty \qquad (3)$$

- Condition (3) is always true when $d = 1$
  - For $d = 2$, (3) holds if and only if $\int_{|x|\le 1} f(x) \log \frac{1}{|x|} dx < \infty$
  - For $d \ge 3$, (3) holds if and only if $\int_{|x|\le 1} f(x) \frac{1}{|x|^{d-2}} dx < \infty$

  In the particular case $f(x) = |x|^{-\beta}$, (3) holds if and only if $0 < \beta < 2$

Hölder continuity of the solution

Using Kolmogorov's continuity theorem (Sanz-Solé and Sarrà 2002) one can prove that:

- For the stochastic heat equation with $d \geq 1$ if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty$$

  for some $\eta \in (0,1)$, then
  - $t \to u(t,x)$ is γ_1-Hölder continuous for $0 < \gamma_1 < \frac{1}{2}(1-\eta)$
  - $x \to u(t,x)$ is γ_2-Hölder continuous for $0 < \gamma_2 < 1-\eta$
- For the stochastic wave wave equation if $d = 1, 2$ similar results hold with $0 < \gamma_1 < \frac{1}{2} \wedge (1-\eta)$
- A different approach based on Sobolev embedding theorems is needed to handle the stochastic wave equation in $d = 3$ (Dalang and Sanz-Solé, Memoirs of AMS 2009)

Regularity of the density

Problem:

- For any fixed (t, x) we want to show that $u(t, x)$ has a density which is infinitely differentiable density with respect to the Lebesgue measure
- Remark: There is no equation for the evolution of the law of $u(t, x)$
- This can be proved using Malliavin Calculus

Malliavin Calculus

- The Malliavin Calculus is a differential calculus on a Gaussian space that was introduced by Malliavin in the 70's to provide a probabilistic proof of Hörmander's hypoellipticity theorem
- By means of an integration-by-parts formula one can derive general formulas for densities of functionals of an underlying Gaussian process, and show their regularity
- The Malliavin Calculus has been applied in a variety of areas:
  - Potential analysis for stochastic partial differential equations (Dalang, Khoshnevisan, Nualart)
  - Computation of Greeks in mathematical finance (Lions, Touzi, Kohatsu-Higa)
  - Ergodicity of stochastic Navier equation (Pardoux, Mattingly)

- Let $\mathcal{S}$ be the class of smooth random variables of the form

$$F = f(W(h_1), \ldots, W(h_n))$$

where $f \in C_b^\infty(\mathbb{R}^n)$, and $h_i \in \mathcal{H}$
- The derivative of F is the $\mathcal{H}$-valued stochastic process

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n)) h_i$$

- The derivative operator D is a closed operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p > 1$

- For any $p > 1$ and for any positive integer k we denote by $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the semi-norm

$$\|F\|_{k,p}^{p} = E(|F|^{p}) + \sum_{j=1}^{k} E\left[\left\|D^{j}F\right\|^{p}\right],$$

  where D^j denotes the iterated derivative
- Set $\mathbb{D}^{\infty} = \cap_{k,p}\mathbb{D}^{k,p}$

- The density $p_F(x)$ of a random variable F can be expressed as

$$p_F(x) = E\left(\mathbf{1}_{\{F<x\}}\delta\left(\frac{DF}{\|DF\|_{\mathcal{H}}^2}\right)\right),$$

where δ is the adjoint of D

- This formula requires $F \in \mathbb{D}^{2,2}$, $E(\|DF\|_{\mathcal{H}}^{-4}) < \infty$, and $DF/\|DF\|_{\mathcal{H}}^2$ in the domain of δ

Criteria for existence and regularity of densities

(I) Bouleau-Hirsch: If $F \in \mathbb{D}^{1,2}$, and $\|DF\|_{\mathcal{H}} > 0$ almost surely, then the probability law of F is absolutely continuous

(II) Malliavin-Watanabe: If $F \in \mathbb{D}^{\infty}$, and $E\left(\|DF\|_{\mathcal{H}}^{-p}\right) < \infty$ for all $p \geq 1$, then F has an infinitely differentiable density

These criteria can be extended to d-dimensional random vectors F replacing $\|DF\|_{\mathcal{H}}$ by the determinant of the Malliavin matrix $\langle DF^i, DF^j \rangle_{\mathcal{H}}$

- Our aim is to apply the criterium (II) to the proof of the regularity of the density of $u(t, x)$

Regularity of the density: Elliptic case

Theorem (N. and Quer-Sardanyons, 2008)

Assume that G satisfies (2), the coefficients σ and b are C^∞ functions with bounded derivatives of all orders and $|\sigma(z)| \geq c > 0$, for all $z \in \mathbb{R}$. Suppose that there exists $\gamma > 0$ such that for all $\delta \in (0, 1]$,

$$g(\delta) := \int_0^\delta \int_{\mathbb{R}^d} |\mathcal{F} G_s(\xi)|^2 \mu(d\xi) ds \geq C\delta^\gamma,$$

for some positive constant C. Then, for all $(t, x) \in (0, \infty) \times \mathbb{R}^d$, the law of $u(t, x)$ has a C^∞ density with respect to the Lebesgue measure on $\mathbb{R}^d$

Sketch of the proof

Let $u(t,x)$ be the solution of Equation (2)

- If b and σ belong to $C_b^\infty(\mathbb{R}^d)$, then $u(t,x) \in \mathbb{D}^\infty$
- Recall that $Du(t,x) \in \mathcal{H} = L^2([0,t];\mathcal{H}_0)$. Set

$$C_{t,x} = \|Du(t,x)\|_{\mathcal{H}}^2 = \int_0^t \|D_s u(t,x)\|_{\mathcal{H}_0}^2 ds$$

To prove the regularity of the density of $u(t,x)$ it suffices to show that

$$E\left(C_{t,x}^{-p}\right) < \infty \qquad (4)$$

for all $t > 0$ and $x \in \mathbb{R}^d$ and $p \geq 2$

Lemma

A nonnegative random variable F satisfies $E(F^{-p}) < \infty$ for all $p \geq 1$ if an only if for all $p \geq 1$ there exists $\epsilon_0 > 0$ such that $P(F < \epsilon) \leq C\epsilon^p$ for all $\epsilon \leq \epsilon_0$

We will apply this lemma to

$$F = C_{t,x} \geq \int_{t-\delta}^{t} \|D_s u(t,x)\|_{\mathcal{H}_0}^2 ds,$$

with a convenient choice of $\delta(\epsilon)$

- Assuming $b = 0$ and applying the operator D to Equation 2 yields

$$\begin{aligned} D_s u(t,x) &= \sigma(u(s,\cdot))G_{t-s}(x,\cdot) \\ &\quad + \int_s^t \int_0^1 G_{t-r}(x,y)\sigma'(u(r,y))D_s u(r,y)W(dy,dr), \end{aligned}$$

if $s < t$ and $D_s u(t,x) = 0$ if $s > t$.

- Fix $\delta > 0$

$$\int_0^t \|D_s u(t,x)\|_{\mathcal{H}_0}^2 ds \geq \frac{1}{2}\int_{t-\delta}^t \|\sigma(u(\theta,\cdot))G_{t-s}(x,\cdot)\|_{\mathcal{H}_0}^2 ds - I_\delta,$$

where

$$I_\delta = \int_{t-\delta}^t \left\| \int_s^t \int_0^1 G_{t-r}(x,y)\sigma'(u(r,y))D_s u(r,y)W(dy,dr) \right\|_{\mathcal{H}_0}^2 ds$$

- We have

$$\int_{t-\delta}^{t} \|\sigma(u(\theta,\cdot))G_{t-s}(x,\cdot)\|_{\mathcal{H}_0}^2 ds \geq c^2 g(\delta)$$

and for any $p \geq 2$

$$E(|I_\delta|^p) \leq C\delta^{p-1} g(\delta)^p$$

- Fix $\epsilon > 0$ and choose $\delta = \delta(\epsilon)$ such that $g(\delta) = \frac{4}{c^2}\epsilon$. Then

$$P\left(\int_0^t \|D_s u(t,x)\|_{\mathcal{H}_0}^2 ds < \epsilon\right) \leq C\epsilon^{\frac{p-1}{\gamma}},$$

which implies the desired result

Nonelliptic degeneracy: Heat equation on $[0, 1]$

Consider the one-dimensional stochastic heat equation driven by a space-time white noise

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x) \qquad (5)$$

- $t \geq 0$, $x \in [0, 1]$, and we impose Dirichlet boundary conditions $u(0, x) = u(1, x) = 0$
- $u(0, x) = u_0(x)$ is continuous, vanishing at 0 and 1
- $\dot{W}(t, x)$ is a space-time white noise

Regularity of the probability law of $u(t,x)$

Theorem (Mueller and N., 2007)

Suppose that $\sigma(u_0(x_0)) \neq 0$ *for some* $x_0 \in (0,1)$, u_0 *is Hölder continuous of order* $\alpha > 0$, *and* σ *and* b *are in* $C_b^\infty(\mathbb{R})$. *Then for each* $t > 0$ *and* $x \in (0,1)$ *the density of* $u(t,x)$ *is infinitely differentiable*

- Pardoux and Zhang, 1993, proved the existence of the density under the same nondegeneracy condition
- Bally and Pardoux, 1998, proved the above result assuming $|\sigma| \geq c > 0$

Sketch of the proof

- The derivative $D_{s,\xi}u(t,x)$ is the solution of the stochastic partial differential equation

$$\frac{\partial D_{s,\xi}u}{\partial t} = \frac{\partial^2 D_{s,\xi}u}{\partial x^2} + b'(u(t,x))D_{s,\xi}u + \sigma'(u(t,x))D_{s,\xi}u\dot{W}(t,x)$$

  on $[s,\infty) \times [0,1]$, with Dirichlet boundary conditions and initial condition $\sigma(u(s,\xi))\delta_0(x-\xi)$

- Also

$$C_{t,x} = \|Du(t,x)\|_{\mathcal{H}}^2 = \int_0^t \int_0^1 |D_{s,\xi}u(t,x)|^2 d\xi ds$$

Suppose that $\sigma(u_0(y)) \geq \delta > 0$ for all $y \in [a,b] \subset (0,1)$. Then, using that $D_{s,\xi}u(t,x) \geq 0$,

$$C_{t,x} \geq \int_0^t \left| \int_a^b D_{s,\xi}u(t,x)d\xi \right|^2 ds$$

Define

$$Y_{t,x}^s = \int_a^b D_{s,\xi}u(t,x)d\xi$$

The random field $\{Y_{t,x}^s, t \geq s, x \in [0,1]\}$ satisfies

$$\frac{\partial Y_{t,x}^s}{\partial t} = b'(u_{t,x})Y_{t,x}^s + \sigma'(u_{t,x})Y_{t,x}^s \dot{W}(t,x)$$

with initial condition

$$Y_{t,x}^s|_{t=s} = \sigma(u_0(x))\mathbf{1}_{[a,b]}(x)$$

Fix $r < 1$ and $\epsilon > 0$ such that $\epsilon^r < t$. Then

$$
\begin{aligned}
P(C_{t,x} < \epsilon) &\leq P\left(\int_0^{\epsilon^r} |(Y_{t,x}^0)^2 - (Y_{t,x}^s)^2| ds > \epsilon\right) \\
&\quad + P\left(Y_{t,x}^0 < \sqrt{2}\epsilon^{\frac{1-r}{2}}\right) \\
&= P(A) + P(B)
\end{aligned}
$$

- By Tchebychev inequality, for any $q \geq 1$

$$
P(A) \leq \epsilon^{(r-1)q} \sup_{0 \leq s \leq \epsilon^r} E(|(Y_{t,x}^0)^2 - (Y_{t,x}^s)^2|^q) \leq C\epsilon^{(r-1)q+\alpha q}
$$

  for some $\alpha > 0$ and it suffices to choose $r > 1 - \alpha$
- The desired estimate for $P(B)$ follows from $E((Y_{t,x}^0)^{-p}) < \infty$ for all $p \geq 1$

Theorem (Mueller and N., 2007)

Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Bu + Hu\dot{W}(t,x),$$

where B and H are bounded and adapted processes, u_0 is not identically zero, and we impose Dirichlet boundary conditions on $[0,1]$. Then, for all $p \geq 1$, $t > 0$ and $x \in (0,1)$,

$$E(u^{-p}(t,x)) < \infty$$

- $Y^0_{t,x}$ satisfies an equation of this type with $B = b'(u)$ and $H = \sigma'(u)$

The proof is based on large deviation estimates (Mueller, 1991): Let Y be a predictable process bounded by $K > 0$. Then for any $\lambda > 0$ and $M > 0$

$$P\left(\sup_{\substack{0\le t\le T\\ |x|\le M}}\left|\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)|Y(s,y)|W(dsdy)\right| > \lambda\right) \le C_1 \exp\left(-\frac{C_2\lambda^2}{\sqrt{T}K^2}\right)$$

$$\frac{\partial u}{\partial t} = \Delta u + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x)$$

- $x \in \mathbb{R}^d$
- $E(\dot{W}(t,x)\dot{W}(s,y)) = \delta_0(t-s)f(x,y)$
- $f(x,y)$ is ρ-Hölder continuous for some $\rho > 0$, $|f(x,y)| \leq C(1+|x|^\gamma + |y|^\beta)$, for some $\gamma, \beta \in [0,2)$, and

$$f(x,y) = \int_{\mathbb{R}^d} h(\xi,x)h(\xi,y)d\xi,$$

where $h(\xi,x)$ has polynomial growth

Theorem (Hu, N. and Song, 2010)

Assume that u_0 is α-Hölder continuous for some $\alpha > 0$, b and σ are in $C_0^\infty(\mathbb{R}^d)$. Suppose that $\sigma(u_0(x_0)) > 0$ and $f(x_0, x_0) > 0$ for some $x_0 \in (0, 1)$, then for each $t > 0$ and $x \in (0, 1)$ the density of $u(t, x)$ is infinitely differentiable

The proof is based on a stochastic version of Feynman-Kac formula for the process

$$V_{s,\xi}(t, x) = \int_{\mathbb{R}^d} h(\xi, y) D_{s,y} u(t, x) dy$$

Further research

1. Prove that the random vector $(u(t, x_1), \ldots, u(t, x_n))$ has a infinitely differentiable density
2. Show that the density of $u(t, x)$ is positive everywhere

Problems 1 and 2 have been solved if $|\sigma| \geq c > 0$ (Bally and Pardoux, 1998), but they are open under the more general nondegeneracy condition $\sigma(u_0(x_0)) \neq 0$

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