Stochastic Geometry and the User Experience in a Wireless Cellular Network

Sayandev Mukherjee
DOCOMO Innovations, Inc.
Palo Alto, CA
Future networks will be heterogeneous

Small(er) cells are an effective way to achieve spectrum reuse
- Micro- and Pico-cells: favored by network operators for offload
- Femto-cells: offloading and no operator backhaul expense
- Relays and Remote Radio Heads: applications to enhance coverage and reduce energy consumption
- Operator-owned D2D transmitters (DII proposal)

Cisco prediction: Overall mobile data traffic is expected to grow to 24.3 exa (10^{18}) bytes per month by 2019, nearly a 10x increase over 2014

Spectrum reuse is a popular and effective way to increase capacity within existing spectrum

Existing macro-cells + new small cells = heterogeneous network (HetNet)
Network densification for greater capacity

Feasible, robust, and relatively cheap: a *dense* network of *small* cells *overlaid* on the existing large-cell (macro-cell) network

Courtesy Prof. Ismail Guvenc, Florida International University, Miami, FL
How to design a ‘good’ HetNet?

• ‘Good’: a chosen set of operational parameters representing a desirable tradeoff between coverage, capacity, and cost

• Coverage and capacity both depend on the distribution of the Signal to Interference plus Noise ratio (SINR) in the network

• Thus, before we can design the HetNet, we need to understand the behavior of the SINR in a HetNet

• Example: considering a picocell overlay on the macrocell network

• How does the distribution of SINR at an arbitrarily-located user (also called the “typical user”) depend on:
  - The density of picocells relative to macrocells?
  - The transmit power of picocells relative to macrocells?
  - The parameters of the wireless channel model?
New approach: combining insights from analysis with targeted simulations

Exhaustive Simulation

• Study any scenario at any desired depth of detail
• Time-consuming to design, code, debug, and run
• Requires a separate simulation for every scenario and for each choice of parameters of the simulation
• Number of combinations of deployment parameters rises exponentially in the number of tiers of the HetNet

Analysis + Targeted Sim.

• Scales to arbitrary numbers of tiers: macro/micro/pico/femto-cells, relays, remote radio heads, D2D transmitters
• Gives overall insights without misinterpreting or being biased by any specific scenario
• Cannot model detailed aspects of PHY layer design
• For exact performance results, need simulation
• Use insights to shrink search space, then run targeted simulations for exact results
Definition of SINR at a “typical user”

- Consider a user at some arbitrary location, set to be the origin
- \( n \) Base Stations (BSs), chosen from the tiers of the HetNet
- Received power at the user from the \( k \)th BS is \( X_k \), \( k = 1, \ldots, n \)
- SINR at user when receiving from the \( k \)th BS is

\[
\Gamma_k = \frac{X_k}{\sum_{j=1}^{n} X_j + Z}
\]

where

\( Z = \text{thermal noise power} + \text{total received power from all BSs in the HetNet other than these } n \text{ BSs} \)
Matrix-vector form of important events

• Recall SINR when receiving from $k$th BS:
  $$\Gamma_k = \frac{X_k}{\sum_{j=1, j \neq k}^{n} X_j + Z}$$

• The **SINR Exceedance Event** (used to calculate joint distributions):
  $$\{\Gamma_1 > \gamma_1, \ldots, \Gamma_n > \gamma_n\}$$
  $$\equiv \left\{ \begin{bmatrix} 1 - \rho_1 & -\rho_1 & \cdots & -\rho_1 \\ -\rho_2 & 1 - \rho_2 & \cdots & -\rho_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_n & \cdots & -\rho_n & 1 - \rho_n \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} > Z \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} \right\}$$
  $$\rho_k = \frac{\gamma_k}{1 + \gamma_k}$$

• **SINR and Power Exceedance Event**: $k \leq n$ (SINR with tier selection bias):
  $$\{\Gamma_1 > \gamma, X_2 > \theta_2 X_1, X_3 > \theta_3 X_1, \ldots, X_k > \theta_k X_1\}$$
  $$\equiv \left\{ \begin{bmatrix} \gamma^{-1} & -1 & \cdots & -1 \\ -\theta_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\theta_k & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} > Z \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$
Canonical SINR event at “typical user”

• Take the event that \( n \) non-negative random variables \( X_1, \ldots, X_n \) belong to the region defined by

\[
\left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{A} \mathbf{x} > \mathbf{\tilde{b}} \right\}
\]

where \( \mathbf{A} \) is an \( n \times n \) real matrix and \( \mathbf{\tilde{b}} \) is an \( n \times 1 \) vector with all entries nonnegative, and the inequality is interpreted to apply component by component.

• Suppose also that all off-diagonal entries in \( \mathbf{A} \) are \( \leq 0 \)

• Such a matrix is called a \( Z \)-matrix

• We also assume that \( \mathbf{A} \) is nonsingular

• For future use, we also define an \( M \)-matrix as one whose inverse has all non-negative entries
The structure of the canonical region

- [Berman&Plemmons’94, Thm. 2.3, Chap. 6 (p. 134), Condition (I.28)]: The canonical region is entirely in the positive orthant if $A$ is an M-matrix, and entirely outside the positive orthant otherwise.
  - Further, if $A$ is an M-matrix, then the canonical region is a cone:
    \[ \left\{ x \in \mathbb{R}^n : Ax > \tilde{b} \right\} = A^{-1}b + \left\{ A^{-1}\beta : \beta > 0 \right\} \]
The probability of the canonical event

- Suppose that $\tilde{b} = Zb$ where $Z$ is a random variable that is independent of $X_1, \ldots, X_n$ and the latter are independent with Erlang Gamma PDFs $f_{X_k}(x)$ for $k = 1, \ldots, n$:

$$
    f_{X_k}(x) = \frac{c_k (c_k x)^{n_k - 1} e^{-c_k x}}{(n_k - 1)!}, \quad x \geq 0, \ n_k \in \{1, 2, \ldots \}
$$

- Let

$$
    F_{A^{-1}}(c) = \int_{\{x \in \mathbb{R}^n : (\exists \beta > 0) x = A^{-1}\beta\}} e^{-c^\top x} \, dx = \det A^{-1} \prod_{k=1}^n \frac{1}{c_k^{\alpha_k(-1)}}
$$

- Then the probability of the canonical event is $(m_k = n_k - 1)$

$$
    \mathbb{P}\{AX > Zb\} = \left[ \prod_{k=1}^n \frac{c_k^{m_k + 1}}{(-1)^{m_k}} \right] \frac{\partial^{m_1 + \ldots + m_n}}{\partial c_1^{m_1} \ldots \partial c_n^{m_n}} \left\{ F_{A^{-1}}(c) \left[ \mathbb{E} e^{-sZ} \bigg|_{s = c^\top A^{-1}b} \right] \right\}
$$
When is a Z-matrix an M-matrix?

• Note that to compute $\mathbb{P}\{AX > Zb\}$ we need to be able to tell when a given Z-matrix $A$ is also an M-matrix

• $[\text{Condition (E}_{17}\text{), Thm. 2.3, Chap. 6, Berman&Plemmons'94}]
  \[ A \text{ is an M-matrix iff every leading principal minor is } > 0: \]
  \[ \det(A([1:k],[1:k])) > 0, \quad k = 1,\ldots,n \quad [\text{Matlab notation}] \]

For the $A$ matrices in our scenarios of interest:
• $\det(A([1:k],[1:k]))$ can be computed in closed form
• $\det(A([1:k],[1:k]))$ is decreasing in $k$
• Thus in our scenarios, $A$ is an M-matrix iff $\det A > 0$
• We will see that we can also calculate $A^{-1}$ in closed form
• Note that $\mathbb{P}\{AX > Zb\} = 0$ if $\det A \leq 0$
What do we need for analytical modeling?

• Important downlink coverage probabilities for heterogeneous cellular networks can be expressed in the canonical form \( P\{Ax > Zb\} \)

• If \( Z, X_1, \ldots, X_n \) are independent nonnegative random variables and \( X_1, \ldots, X_n \) are Erlang distributed, then the probability \( P\{Ax > Zb\} \) can be calculated in closed form for \( A \) a \( Z \)-matrix and all-positive \( b \) if
  – \( A \) is an \( M \)-matrix; and
  – the Laplace Transform of \( Z \) can be calculated in closed form

• So we need a model for placement of BSs that:
  – is applicable to real-world network deployments; and
  – allows for the Laplace Transform of \( Z \) to be calculated analytically
What about the ideal hexagonal lattice?

- Dubious applicability to the “real world”
- Cannot compute Laplace Transform of interference analytically
- Same problem for any other finite regular deployment of BSs, but …

From Blaszczyszyn et al., 2010
Asymptotically, finite BS layouts “are” PPP

• For any arbitrary layout of a tier of BSs in the plane with iid lognormal fading on all links to a user located at the origin:
  – If the area-average number of BSs is finite and the standard deviation of the lognormal fading is large enough [Blaszczyzyn, Karray and Keeler, Infocom 2013]
    • The point process of link losses at the user terminal converges in distribution to the point process of link losses from a tier of BSs whose locations are the points of a Poisson Point Process (PPP)
    • The above holds regardless of the link path loss model
  – Generalized to iid Suzuki fading (lognormal shadowing and Rayleigh fast fading) [Blaszczyzyn, Karray and Keeler, T-WC 2015]
  – Further generalized to arbitrary iid fading satisfying general and mild conditions [Keeler, Ross, and Xia, ArXiv 1411.3757, Nov. 2014]
• The above are asymptotic results. What happens in “real life”? 
**PPP vs Hex Lattice: Same Trends**

**Simulation results**

- **Hex lattice**
- **PPP**

**Key Model Assumption:** BS locations in tiers of HetNet modeled by independent Poisson Point Processes (PPP)

**Regular placement imposes minimum inter-site distance**

**PPP does not, so interference is expected to be stronger than for regular placement**

**Insights into the behavior of any practical deployment can be obtained by a combination of the two curves**

- Note the near-constant 3dB gap; also suggested from theory*

Link loss and received power

• Consider a single tier $i$, say, and a user at the origin
• BSs in this tier are located at points of homogeneous PPP $\Phi_i$
• The link loss on the link between BS $b \in \Phi_i$ and this user is

\[ L_b = \frac{R_b^{\alpha_i}}{K_i H_b} \]

- Here, $\alpha_i > 2$, the path-loss exponent, is the slope and $K_i$ is the intercept of the slope-intercept path-loss model
- $R_b$ is the distance between BS $b$ and the user at the origin
- $H_b$ is the fading coefficient on the link (assumed Erlang-Gamma distributed with unit mean)

• The received power at the user from BS $b \in \Phi_i$ is

\[ Y_b = \frac{P_i K_i H_b}{R_b^{\alpha_i}} \]

Distance of $b$ from user
Fading Coefficient
Constant; depends on geometry
Transmit power: same across all BSs in a tier
Analytical tractability of PPP modeling

- For an arbitrarily-located user in a tier $i$ of BSs whose locations are the points of a homogeneous PPP $\Phi_i$, with *arbitrary* iid fading to the user:
  - The point process $\{L_b: b \in \Phi_i\}$ of link losses at this user terminal from all BSs in the tier is a non-homogeneous PPP whose intensity measure can be obtained analytically as [Haenggi, 2008; Madhusudhanan et al., 2009; Blaszczyszyn et al., 2010]:
    \[
    \Lambda_L(dl) = \left(\frac{2\pi}{\alpha_i}\right) K_i^{-2/\alpha_i} \left(\lambda_i E[H_i^{2/\alpha_i}]\right) l^{-\left(1-2/\alpha_i\right)} \, dl, \quad l > 0
    \]
  - Here $H_i$ is a random variable with the common distribution of the iid fading coefficients $\{H_b: b \in \Phi_i\}$; recall that $E[H_i] = 1$
  - $\lambda_i$ is the (constant) intensity of the homogeneous PPP $\Phi_i$
- This link-loss point process has the same intensity function as the path-loss (i.e., with *no fading*) point process from a different homogeneous PPP-located tier of BSs whose density is that of the original tier of BSs times a moment of the fading coefficient $H_i$
  - Of course, we do require that $E[H_i^{2/\alpha_i}] < \infty$
- What about the set of received powers $\{Y_b: b \in \Phi_i\}$?
Analytical tractability of PPP modeling (contd.)

• Formally applying the Mapping Theorem, the intensity measure of $P_i$ times the inverse of the link losses is

$$
\Lambda_Y(dy) = \frac{\pi \lambda_i \delta_i (P_i K_i)^{\delta_i} \mathbb{E}[H_i^{\delta_i}]}{y^{1+\delta_i}} \, dy, \quad y > 0, \quad \delta_i = \frac{2}{\alpha_i} \in (0, 1)
$$

  - This should be the intensity measure of $\{Y_b: b \in \Phi_i\}$, but unfortunately, $\Lambda_Y([0, y]) = \infty$ for all $y > 0$, though $\Lambda_Y(K) < \infty$ for all compact $K \subset (0, \infty)$

• The set $\{Y_b: b \in \Phi_i\}$ of received powers is not a point process on the space $[0, \infty)$ because it has a limit point at 0

  - Fortunately, $\{0\} \cup$ [the PPP on $(0, \infty)$ with intensity measure $\Lambda_Y$] is a Poisson random set [Molchanov, 2005, p. 109] and has the same capacity functional as the PPP on $(0, \infty)$ with intensity measure $\Lambda_Y$

  - Received power is never 0, so we identify $\{Y_b: b \in \Phi_i\}$ with the PPP on $(0, \infty)$ with intensity measure $\Lambda_Y(dy) = (C/y^{1+\delta}) \, dy$

  - Consequence: The signal to signal-plus-interference ratio, called signal to total interference ratio or STIR, is a two-parameter Poisson-Dirichlet point process [Keeler and Blaszczyszyn, Wireless Comm. Letts., Oct. 2014]
Cell association in a cellular network

- In LTE, all BSs transmit *reference symbols (RSs)* that identify them.
- A user terminal aggregates multiple measurements of the received reference symbol signals and smooths out fast fading.
- Serving BS(s) are selected on the basis of the smoothed reference symbol STIR, called reference symbol received quality or RSRQ.
- Assume residual fading is slow only, iid across all links to the user.
- Then, for RSRQ distribution, the BS deployment is equivalent to another homogeneous PPP without fading and an adjusted density.
  - In each tier, the $n$ “strongest” BSs (original deployment) → the $n$ nearest BSs.
  - For coordinated multipoint (CoMP) transmissions from BSs in a given tier $i$, we select the $n$ “strongest” BSs (original deployment) where the $n$th-strongest is no more than, say, $r_i$ dB weaker in RSRQ than the strongest → conditions on the distances of the $n$ nearest BSs.
  - For selection of serving BS(s) across tiers of a HetNet, the RSRQs from the candidate serving BS(s) in each tier $i$ are weighted by *tier selection bias* factors, then compared across all tiers.
Coverage in a cellular network

- Consider an arbitrarily-located user (a “typical user”) in a cellular network (possibly with multiple tiers)
- The coverage probability for this user is the probability that the RSRQ (from its set of serving BS(s) selected across all tiers) exceeds some threshold (which may be dependent upon the tiers of the serving BSs)
  - The joint distribution of the top $n$ ordered RSRQs from each tier is available analytically from the fact that the STIRs form a two-parameter Poisson-Dirichlet point process [Handa, *Bernoulli* vol. 15, no. 4, pp. 1082-1116, 2009]
- Note that coverage is defined in terms of received STIR of the reference symbols, not data symbols
- In other words, coverage just means that the user can maintain radio contact with the selected serving BS(s), but says nothing about the bit rate of data traffic on such links
Instantaneous (data) SINR in a cellular network

- Consider an arbitrarily-located user (a “typical user”) in a cellular network (possibly with multiple tiers)
- The data SINR is the instantaneous SINR at this user when it is receiving data (not reference symbols) from the serving BS(s) selected on the basis of (smoothed) RSRQ
  - Note that candidate serving BSs from each tier were selected on the basis of geometric proximity to the user (in the equivalent network with no fading)
  - However, even in this equivalent network, when the selected serving BS(s) transmit data to the user, the instantaneous SINR now includes the fast fading on the links between these BSs and the user
- Example: Candidate serving BS in each tier is the (single) nearest BS (to the user) in that tier; $n$ tiers; the single serving BS is the instantaneous strongest one (without tier selection bias); instantaneous SINR distribution is

$$\mathbb{P}\{\Gamma > \gamma\} = \mathbb{P}\left(\bigcup_{i=1}^{n}\{\Gamma_i > \gamma\}\right) = \sum_{k=1}^{n} (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} \mathbb{P}\{\Gamma_{i_j} > \gamma, j = 1, \ldots, k\}$$

SINR Exceedance Event
Recall

- The SINR exceedance event can be expressed as $\mathbb{P}\{AX > Zb\}$
- If $Z, X_1,...,X_n$ are independent nonnegative random variables and $X_1,...,X_n$ are Erlang distributed, then the probability $\mathbb{P}\{AX > Zb\}$ can be calculated in closed form for $A$ a $Z$-matrix and all-positive $b$ if
  - $A$ is an $M$-matrix; and
  - the Laplace Transform of $Z$ can be calculated in closed form
- We shall see that:
  - The corresponding Laplace Transform for $Z$ can be calculated in closed form when base station locations are modeled by PPPs but not when they are placed on a regular hexagonal lattice, say
  - Compared to regular hexagonal lattice located BS placement, interference power is always higher for PPP placement
    - Reason: no minimum distance requirement between BSs in PPP model
  - Thus, PPP model yields conservative results for coverage
Single tier: Laplace Transform of $Z$

- Consider a single tier $i$, say
- BSs in this tier are located at points of homogeneous PPP $\Phi_i$
- Received power at user from BS $b \in \Phi_i$ is $Y_b = \frac{P_i K_i H_b}{R_b^{\alpha_i}}$

- Want Laplace Transform of total received power at user from all BSs in tier $i$ that are at least at distance $d$ from the user:

\[
\mathbb{E} \exp \left[ -s \sum_{b \in \Phi_i: R_b > d} Y_b \right] = \mathbb{E}_{\Phi_i} \left[ \prod_{b \in \Phi_i} g_s(b) \right]
\]

\[
= \exp \left\{ -2\pi \lambda_i \int_0^{\infty} r \left[ 1 - \mathcal{L}_H \left( \frac{sP_i K_i}{r^{\alpha_i}} 1\{r > d\} \right) \right] dr \right\}.
\]

$g_s(b) = \mathbb{E}_{H_b} \left[ \exp \left( -sP_i K_i \frac{H_b}{R_b^{\alpha_i}} 1\{R_b > d\} \right) \right] = \mathcal{L}_H \left( \frac{sP_i K_i}{R_b^{\alpha_i}} 1\{R_b > d\} \right)$

Probability Generating Functional
Example 1: joint distribution of SIR from tiers

- Assume i.i.d. Rayleigh fading on all links, no noise, all \( n \) tiers open
- Assume candidate serving BSs for the user are the nearest BSs in the \( n \) tiers, whose distances \( r_k \) from the user are known
- **Conditioned on these distances**, the received power \( X_k \) at user from the candidate serving BS in tier \( k \) is Exponential with mean \( 1/c_k \)

\[
c_k = \frac{r_k^{\alpha_k}}{(P_k K_k)}, \quad k = 1, \ldots, n
\]

- Let the SIR at the user if it is served by the \( k \)th candidate serving BS be \( \Gamma_k \). Then the joint distribution of these SIRs given \( \{r_k\} \) is

\[
P\{\Gamma_1 > \gamma_1, \ldots, \Gamma_n > \gamma_n\} = P\{AX > Z\rho\} = F_{A^{-1}(c)}(c^\top A^{-1}\rho)
\]

\[
\mathcal{A} = \begin{bmatrix}
1 - \rho_1 & -\rho_1 & \cdots & -\rho_1 \\
-\rho_2 & 1 - \rho_2 & \cdots & -\rho_2 \\
\vdots & \vdots & \ddots & \vdots \\
-\rho_n & \cdots & -\rho_n & 1 - \rho_n
\end{bmatrix} = I_n - \rho 1_n^\top, \quad \rho = \begin{bmatrix}
\gamma_1/(1 + \gamma_1) \\
\gamma_2/(1 + \gamma_2) \\
\vdots \\
\gamma_n/(1 + \gamma_n)
\end{bmatrix}
\]

\[
\mathcal{L}_Z(s) = \exp\left\{-2\pi \sum_{k=1}^{n} \frac{\lambda_k r_k^{2\alpha_k}}{(\alpha_k - 2)[1 + r_k^{\alpha_k}/(s P_k K_k)]} \right\}
\]

\[
\left\{2F_1 \left(1, 1; 2 - \frac{2}{\alpha_k}; 1 + \frac{1}{r_k^{\alpha_k}/(s P_k K_k)} \right)\right\}
\]

5/18/15
Example 2: single tier, $X_1$ Erlang-Gamma

- Single-tier, no noise, serving BS is nearest one to user at distance $r_1$
- *Conditioned on this distance*, the received power $X_1$ at user from the serving BS is Erlang-Gamma:
  \[
  f_{X_1}(x) = \frac{c^m e^{-cx}}{m!}, \quad x \geq 0, \quad m \in \{0, 1, 2, \ldots\}, \quad c = r_1^\alpha/(PK)
  \]
- From the expressions for the probability of the canonical event, the distribution of SIR is given by
  \[
  \mathbb{P}\{\Gamma > \gamma | r_1\} = \frac{c^{m+1}}{(-1)^m} \frac{\partial^m}{\partial c^m} \mathcal{L}_Z(\gamma/c)
  \]
  where
  \[
  \mathcal{L}_Z(s) = \exp\left\{-2\pi \frac{\lambda r_1^2}{(\alpha - 2)[1 + r_1^\alpha/(sPK)]} \right\}^{2F_1}\left(1, 1; 2 - \frac{2}{\alpha}; 1 + \frac{1}{1 + r_1^\alpha/(sPK)}\right)
  \]
- [Li, Zhang, and Ben Letaief, T-WC May 2014] show that
  \[
  \mathbb{P}\{\Gamma > \gamma\} = \max_{0 \leq j \leq m} \sum_{i=0}^{m} |T_{ij}|
  \]
  where $T = [T_{ij}]_{i,j=0,\ldots,m}$ is a certain lower-triangular Toeplitz matrix
Example 3: tier selection biases –
Instantaneous SINR distribution

- i.i.d. Rayleigh fading on all links, no noise, candidate serving BSs for the user are the nearest BSs, at distances $r_i$ from the user
- Conditioned on these distances, the received power $X_i$ at the user from the candidate serving BS in tier $i$ is exponential with mean $1/c_i$
  \[ c_i = r_i^{\alpha_i} / (P_i K_i), \quad i = 1, \ldots, n \]
- The serving tier is $I = \arg \max_{1 \leq i \leq n} \tau_i X_i$ where $\{\tau_i\}$ are biases
- Let the SINR at the user if it is served by the $i$th candidate serving BS be $\Gamma_i$. Then given $\{r_i\}$ the actual SINR at the user is $\Gamma = \Gamma_I$
- The SINR distribution is
  \[ P\{\Gamma > \gamma\} = \sum_{i=1}^{n} P\{\Gamma_i > \gamma, (\forall j \neq i) \tau_j X_j \leq \tau_i X_i\} \]
- Written out in terms of SINR and Power Exceedance Event:
  \[ P\{\Gamma_i > \gamma, (\forall j \neq i) \tau_j X_j \leq \tau_i X_i\} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{i_1, \ldots, i_k} P\left\{ \frac{\Gamma_{i_1}}{\tau_{i_1}} X_{i_1}, \ldots, \frac{\tau_i}{\tau_{i_k}} X_{i_k} > \Gamma_i \right\} \]
Example 3: tier selection biases – Instantaneous SINR distribution (contd.)

- The SINR and Power Exceedance Event probability given \( \{r_i\} \) is

\[
\mathbb{P}\{ \Gamma_1 > \gamma, X_2 > \theta_2 X_1, \ldots, X_k > \theta_k X_1 \} = \mathbb{P}\{AX > Ze_1\} = F_{A^{-1}}(c) \mathcal{L}_Z (c^\top A^{-1} e_1)
\]

\[
A = \begin{bmatrix}
\gamma^{-1} & -1 & \cdots & -1 \\
-\theta_2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\theta_k & \cdots & 0 & 1
\end{bmatrix} = I + [(1 + \gamma^{-1})e_1 - \theta]e_1^\top - e_1 1^\top, \quad \theta = \begin{bmatrix}
1 \\
\theta_2 \\
\vdots \\
\theta_k
\end{bmatrix}
\]

\[
\det A = \gamma^{-1} - \theta_2 - \cdots - \theta_k
\]

\[
A^{-1} = I + \frac{1 + \theta_2 + \cdots + \theta_k}{\gamma^{-1} - \theta_2 - \cdots - \theta_k} - e_1 e_1^\top
\]

- Note: above probability is 0 if \( \det A \leq 0 \)
- Recall: from PPP theory, Laplace Transform of \( Z \) is known to be

\[
\mathcal{L}_Z(s) = \exp \left\{ -2\pi \sum_{k=1}^{n} \frac{\lambda_k r_k^2}{(\alpha_k - 2)[1 + r_k^{\alpha_k}/(s P_k K_k)]} \right\} \Gamma_1 x^{\alpha_k - 1} 2F_1 \left( 1, 1; 2 - \frac{2}{\alpha_k}; 1 + r_k^{\alpha_k}/(s P_k K_k) \right)
\]

5/18/15
Example 4: CoMP with Interference Alignment -- Instantaneous SINR distribution

- Single tier, i.i.d. Rayleigh fading on all links, no noise, the cluster of transmitting BSs for the user are the $M$ nearest BSs, at distances $r_1 < \ldots < r_M$ from the user, and the serving BS is the nearest one
- Transmitters (BSs) and receiver (user) have multiple antennas transmitting a total of $d$ streams, say, on each link
- Interference Alignment (IA) ensures a $d$-dimensional channel that is free of in-cluster interference – all interference is from BSs farther away than $r_M$
- Conditioned on $r_1, \ldots, r_M$, the received power $X^{(l)}$ on the $l$th stream ($l = 1, \ldots, d$) is Exponential with mean $d r_1^\alpha / (PK)$
- The distribution of the SIR $\Gamma^{(l)}$ on the $l$th stream is therefore

$$
P\{\Gamma^{(l)} > \gamma\} = P\{X^{(l)} > \gamma Z^{(l)}\} = \mathcal{L}_{Z^{(l)}}(\gamma d r_1^\alpha)$$

where

$$
\mathcal{L}_{Z^{(l)}}(s) = \exp\left\{-2\pi \frac{\lambda r_M^2}{(\alpha - 2)[1 + r_M^\alpha/(sPK)]} \, 2F_1 \left(1, 1; 2 - \frac{2}{\alpha}; \frac{1}{1 + r_M^\alpha/(sPK)}\right)\right\}
$$
Example 5: Instantaneous switching – maximum instantaneous SIR distribution

- Now let the user be served by the BS that is received with *maximum power* at that instant (i.e., with *maximum instantaneous SIR*).
- For a single-tier deployment, the relationship between STIR and the two-parameter Poisson-Dirichlet process yields the PDF of the maximum STIR (and therefore also the maximum SIR).
- For more than one tier, let $\Phi$ be the superposition PPP over all tiers. When receiving from any BS $b$ in $\Phi$, if the received power at the user is $Y_b$, the SINR is 
  $$\hat{\Gamma}_b = \frac{Y_b}{\sum_{b' \in \Phi \setminus \{b\}} Y_{b'}}$$

- The distribution of maximum instantaneous SIR is thus:
  $$\mathbb{P} \left( \bigcup_{k=1}^{n} \bigcup_{b \in \Phi_k} \{\hat{\Gamma}_b > \gamma\} \right) = \sum_{m=1}^{n} \sum_{n_1, \ldots, n_m \in \mathbb{N}} \sum_{1 \leq k_1 < \cdots < k_m \leq n} \left( -1 \right)^{n_1 + \cdots + n_m - 1} \frac{1}{n_1! \cdots n_m!} \sum_{b_1, \ldots, b_{n_1}, \ldots, b_{n_m} \in \Phi_{k_1}, \ldots, \Phi_{k_m}} \mathbb{P}\{\hat{\Gamma}_{b_j} > \gamma, j = 1, \ldots, n_1 + \cdots + n_m\}$$
  
  $\mathbb{P}\{\hat{\Gamma}_{b_j} > \gamma, j = 1, \ldots, n_1 + \cdots + n_m\}$
  
  $\mathbb{P}\{\hat{\Gamma}_{b_j} > \gamma, j = 1, \ldots, n_1 + \cdots + n_m\}$
  
  $\mathbb{P}\{\hat{\Gamma}_{b_j} > \gamma, j = 1, \ldots, n_1 + \cdots + n_m\}$
  
  $\mathbb{P}\{\hat{\Gamma}_{b_j} > \gamma, j = 1, \ldots, n_1 + \cdots + n_m\}$
Example 5 (contd.): Computing the terms

- For each tier $\Phi_k$, the iid fades may be assumed Rayleigh after appropriately adjusting the density of the BS PPP for that tier. Then the set of received powers $\Psi_k = \{Y_b: b \in \Phi_k\}$ is identified with the non-homogeneous 1-D PPP with intensity function

$$\nu_k(y) = 2\pi \lambda_k \int_0^\infty r f_X(y) \, dr, \quad y \geq 0, \quad X \sim \text{Exp}(P_k K_k / r^{\alpha_k})$$

- Then over all sets of $q = n_1 + \ldots + n_m$ BSs we have

$$\sum_{b_1, \ldots, b_{n_1} \in \Phi_{k_1}, \ldots, b_{q-n_m+1}, \ldots, b_q \in \Phi_{k_m}} \mathbb{P}\{\Gamma_{b_j} > \gamma, \ j = 1, \ldots, q\}$$

$$= \mathbb{E}\left[ \sum_{y_1, \ldots, y_q \in \Psi} 1 \left\{ \frac{\sum_{l=1}^q y_l + \sum_{y \in \Psi \setminus \{y_1, \ldots, y_q\}} y}{y_j} > \gamma, \ j = 1, \ldots, q \right\} \right]$$

$$= \int_0^\infty dx_1 \nu^{(1)}(x_1) \cdots \int_0^\infty dx_q \nu^{(q)}(x_q) \mathbb{E}_\Psi \left[ 1 \left\{ \frac{\sum_{l=1}^q x_l + \sum_{y \in \Psi} y}{y_j} > \gamma, \ j = 1, \ldots, q \right\} \right]$$

$$= (2\pi)^q \left( \prod_{l=1}^m \lambda_{kl}^{n_l} \right) \int_0^\infty dr_1 \cdots \int_0^\infty dr_q \mathbb{P}\{A \mathbf{X} > W \rho\} \quad W = \sum_{y \in \Psi} y = \sum_{b \in \Phi} Y_b$$
Example 5 (contd.): Final expressions

- Note that $P\{AX > W\rho\} = 0$ if $\det A \leq 0$
  \[ A = I_q - \rho 1_q^\top \Rightarrow \det A = 1 - 1_q^\top \rho = 1 - \frac{\gamma}{1 + \gamma} \sum_{l=1}^{m} n_l \]

- So $P\{AX > W\rho\} > 0$ for only finitely many $m, n_1, \ldots, n_m$

- Suppose $\alpha_1 = \ldots = \alpha_n = \alpha$

- If $\gamma > 1$, then $P\{AX > W\rho\} > 0$ only for $m = 1, n_1 = 1$
  - Then we obtain the elegant closed-form result [Dhillon et al., JSAC'12]
  \[ P \left( \bigcup_{k=1}^{n} \bigcup_{b \in \Phi_k} \{ \tilde{T}_b > \gamma \} \right) = \frac{\text{sinc} (2/\alpha)}{\gamma^{2/\alpha}} \]

- If $\gamma > 0$, and $n \geq \text{ceil}(1/\gamma)$, we have [Mukherjee, 2014]
  \[ P \left( \bigcup_{k=1}^{n} \bigcup_{b \in \Phi_k} \{ \tilde{T}_b > \gamma \} \right) = \sum_{l=1}^{[\gamma^{-1}]} \left( \frac{(-1)^{l-1} \text{sinc}(2/\alpha)^l}{l[1 - l\gamma/(1 + \gamma)]} \left( \frac{1}{\gamma} - (l-1) \right) \right)^{2l/\alpha} T(l) \]

\[ T(l) = \begin{cases} 1/(1 + \gamma), & l = 1, \\ \frac{8\sqrt{\pi} \Gamma(2/\alpha)}{4^{2/\alpha} \Gamma(3/2 + 2/\alpha)} \left( \frac{1-\gamma}{1+\gamma} \right)^2 2F_1 \left( \frac{1}{2}, 1; \frac{3}{2} + \frac{2}{\alpha}; \frac{(1-\gamma)}{1+\gamma}^2 \right), & l = 2 \end{cases} \]
Example 6: coverage probability in a multi-tier HetNet

- The serving BS is the one that is received with maximum RSRQ at the user (i.e., with maximum STIR, equivalent to maximum SIR)
- Let Φ be the PPP of all BS locations. When receiving from any BS $b$ in Φ, if the received power at the user is $Y_b$, the SIR is
  \[ \tilde{I}_b = \frac{Y_b}{\sum_{b' \in \Phi \setminus \{b\}} Y_{b'}} \]
- Suppose the SIR threshold for coverage by tier $k$ is $\beta_k$
- The probability of coverage is thus
  \[
  \mathbb{P}\left( \bigcup_{k=1}^{n} \bigcup_{b \in \Phi_k} \{ \tilde{I}_b > \beta_k \} \right) = \sum_{m=1}^{n} \sum_{n_1, \ldots, n_m \in \mathbb{N}} (-1)^{n_1+\ldots+n_m-1} \frac{1}{n_1! \cdots n_m!} \sum_{1 \leq k_1 < \cdots < k_m \leq n} \mathbb{P}\{ \tilde{I}_{b_{j}} > \beta_{k(j)}, j = 1, \ldots, n_1 + \cdots + n_m \}
  \]
  \[ k(j) = \sum_{l=1}^{m} k_l \left( n_1 + \cdots + n_{l-1} + 1 \leq j \leq n_1 + \cdots + n_l \right) \]
Example 6 (contd.): coverage probability

- Note that $P\{\mathbf{AX} > W\mathbf{\rho}\} = 0$ if $\det \mathbf{A} \leq 0$

$$
\mathbf{A} = \mathbf{I}_q - \mathbf{\rho} \mathbf{1}_q^T \Rightarrow \det \mathbf{A} = 1 - (\mathbf{1}_q^T \mathbf{\rho}) = 1 - \sum_{l=1}^{m} n_l \frac{\beta_{kl}}{1 + \beta_{kl}}
$$

- So $P\{\mathbf{AX} > W\mathbf{\rho}\} > 0$ for only finitely many $m, n_1, \ldots, n_m$

- Suppose $\alpha_1 = \ldots = \alpha_n = \alpha$

- If $\beta_1, \ldots, \beta_n > 1$, then $P\{\mathbf{AX} > W\mathbf{\rho}\} > 0$ only for $m = 1, n_1 = 1$
  - Then [Dhillon et al., JSAC’12]

$$
P\left( \bigcup_{k=1}^{n} \bigcup_{b \in \Phi_k} \{ \tilde{I}_b > \beta_k \} \right) = \frac{\text{sinc}(2/\alpha)}{\sum_{j=1}^{n} \lambda_j (P_j K_j)^{2/\alpha}} \sum_{k=1}^{n} \lambda_k \left( \frac{P_k K_k}{\beta_k} \right)^{2/\alpha}
$$

$$
P\left( \bigcup_{k=1}^{n} \bigcup_{b \in \Phi_k} \{ \tilde{I}_b > \beta_k \} \right) = \mathbb{E}_I P\{ I > \beta_I | I \} = \mathbb{P}\{ I = k \} = \frac{\lambda_k (P_k K_k)^{2/\alpha}}{\lambda_1 (P_1 K_1)^{2/\alpha} + \ldots + \lambda_n (P_n K_n)^{2/\alpha}}, \quad k = 1, \ldots, n
$$

$P\{ I = k \} = \frac{\lambda_k (P_k K_k)^{2/\alpha}}{\lambda_1 (P_1 K_1)^{2/\alpha} + \ldots + \lambda_n (P_n K_n)^{2/\alpha}}$

= Probability that a randomly-selected BS belongs to tier $k$ in an equivalent overall network with density $\lambda = \sum_{k=1}^{n} \lambda_k (P_k K_k)^{2/\alpha}$ and $P = 1, K = 1$ for all BSs
What about non-Erlang distributed $X_k$?

- [DeVore&Lorentz’93, Problem 5.6, p. 14] An arbitrary PDF can be uniformly approximated by an infinite mixture of Erlang Gamma densities:

  If the function $f$ is continuous on $[0, \infty)$ and has limit zero for $x \to +\infty$,

  \[
  \lim_{u \to \infty} S_u(x) = f(x) \text{ uniformly for } 0 \leq x < +\infty,
  \]

  where $S_u(x)$ is defined by

  \[
  S_u(x) = e^{-ux} \sum_{k=0}^{\infty} \frac{(ux)^k}{k!} f \left( \frac{k}{u} \right), \quad u > 0.
  \]

- $X_1, \ldots, X_n$ are the received powers over wireless links
  - For Nakagami fast fading with lognormal slow fading, finite Gamma (but not necessarily Erlang) mixture approximations to the PDF of $X_k$ have been provided in [Atapattu et al., T-WC’11]
  - Convert to finite Erlang Gamma mixture using the method in eqns. (16)-(17) of [Almhana et al., ICC’06]
Data rates in a cellular network

- Conditioned on the distances to the BSs and the fades on the links from those BSs to the user, the achievable data rate to the user is given by Shannon’s formula: 
  \[ C = \log_2(1 + \Gamma) \text{ bits/s/Hz} \]
- Here \( \Gamma \) is the instantaneous data rate to the user.
- The mean achievable data rate to the user is therefore:
  \[
  \mathbb{E}[C] = \int_0^\infty \mathbb{P}\{C > x\} \, dx = \frac{1}{\ln 2} \int_0^\infty \mathbb{P}\{\Gamma > \gamma\} \, d\gamma
  \]
- This is a measure of the **Service Quality as experienced by the user**:
  - User is interested only in his/her own data rate, not that of others.
  - The above is the expected achievable data rate for a “typical user”.
  - Says nothing about the service quality experienced by many simultaneous users of the network, which is what the network operator is interested in.
  - Need to define the appropriate metric from the network operator’s PoV.
Data rates in a cellular network (contd.)

- The network operator is interested in the mean achievable total throughput to all users served in a typical cell (bits/s/Hz).
- Single-tier PPP with density $\lambda$: mean area of the typical cell $= 1/\lambda$.
- Average throughput per unit area $= \lambda \times$ the mean achievable total throughput to all users served in a typical cell (bits/s/Hz/km$^2$).
- Recall: users are served by nearest BSs in an equivalent PPP.
- So: typical cell is that in the Poisson-Voronoi tessellation.
  - Where are the users in this typical cell?
  - How does the BS decide which of these users to serve at a given instant?
- Simplify: assume user locations given by independent PPP, so
  - User locations are iid uniformly distributed over the typical cell.
  - BS simply cycles through these users serving one at a time (round-robin).
- Total throughput to users in typical cell $= \text{throughput to a single user whose location is uniformly distributed over the typical cell}$.
- Distribution of distance between user and nucleus of typical cell?
Distribution of BS-user distance in typical cell

- Recall -- for a typical user: Distance $D$ from arbitrarily-located user to nearest BS: $\Pr\{D > r\} = \exp(-\pi \lambda r^2)$  [Distance in zero cell]
- Distance $R_{b',*}$ from typical cell nucleus to randomly-selected served user: $\Pr\{R_{b',*} > r\} = \exp(-c \pi \lambda r^2)$, where $c = 1.25$ (from empirical fit)

We can continue to use the previously-derived results on SINR and data rate for a typical user with the “adjusted” BS density $c\lambda$

Yu, Yang, Ishii, Mukherjee, JSAC 2015
Is $E[\log_2(1 + \Gamma)]$ the ergodic rate to a user?

- No, because $\Gamma$ is the instantaneous SINR (for a system snapshot)
- The ergodic rate by definition requires transmissions to be long enough to encounter every state of the channel
- For ergodic rate: average over all states of the channel to the user --
  - Consider single tier, homogeneous PPP located BSs
  - BSs transmit iid complex symbols
  - Baseband equivalent complex total interference signal at any user location from all BSs more than some distance away is not Gaussian [Gulati et al., T-SP 2010, Guan and Di Renzo, T-COM 2014]
  - Verified by simulation [Aljuaid and Yanikomeroglu, T-VT 2010]
- However, using a codebook and decoder designed for Gaussian noise, the spectral efficiency is as if the interference were Gaussian! [Lapidoth and Shamai, T-IT 2002]
- Then use $E [\log_2(1 + \text{SIR})]$ for ergodic rate by approximating interference by CAWGN with the same covariance matrix [Mungara et al., 2015]
Conclusions

• Number and variety of deployment scenarios for HetNets too great for detailed simulation study of each and every one of them

• PPP location model for BSs in the tiers of a HetNet
  – Is mathematically tractable and scalable to arbitrary numbers of tiers
  – Yields important insights into SINR distributions throughout the network, e.g., coverage in a single-tier network is not dependent on BS density
  – For important special cases of practical interest, closed-form results are obtainable for coverage
  – Compared to the ‘classical’ regular hexagonal BS location model, the PPP model yields conservative results for both coverage probability and ergodic rate to arbitrarily-located users

• The zero-cell versus typical-cell calculations illustrate the quality of the experience from the perspectives of the user and the network operator, respectively

• Substantial additional approximate results available by relaxing certain assumptions
References

• B. Błaszczyszyn, M.K. Karray, F.X. Klepper, “Impact of the geometry, path-loss exponent and random shadowing on the mean interference factor in wireless cellular networks,” *Proc. WMNC* 2010
References (contd.)