Mathematical tools for analysis, modeling and simulation of spatial networks on various length scales

Part I

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Contents

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- **Aim:** Stochastic modeling of networks for
  - description of networks by only a few parameters

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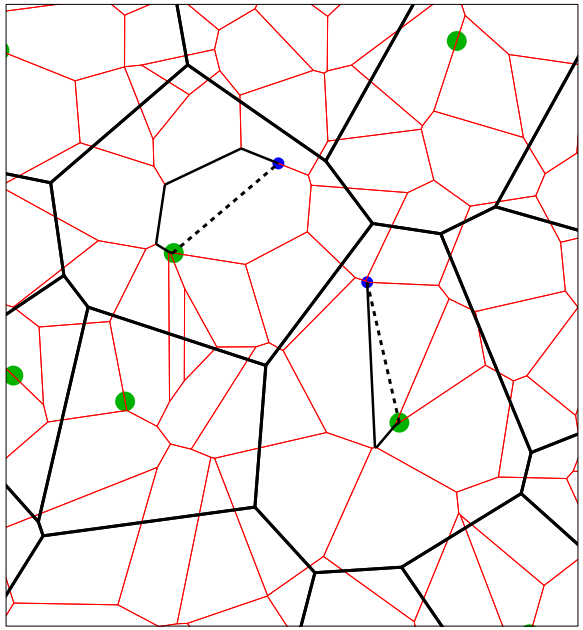

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- $(\Omega, \mathcal{A}, \mathbb{P})$ *some probability space,*
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- Henceforth: Identify $X = \sum_{n=1}^{\infty} \delta_{X_n}$ and $\{X_n\}$, where we write $X = \{X_n\}$
- In other words: The point process $X = \{X_n\}$ is identified with the random counting measure $X = \sum_{n=1}^{\infty} \delta_{X_n}$.

Intensity measure, stationarity and Palm distribution

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- *the intensity measure $\mu : \mathcal{B}(\mathbb{R}^2) \to [0, \infty]$ of X is defined as*

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- *Note that a point process X^o with distribution P_X^o is called a Palm version of X.*

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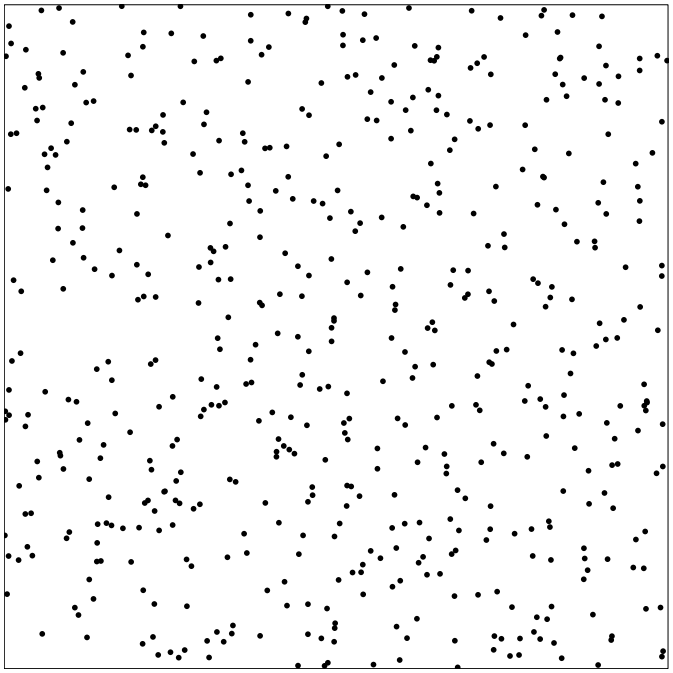

Realization of a stationary Poisson process

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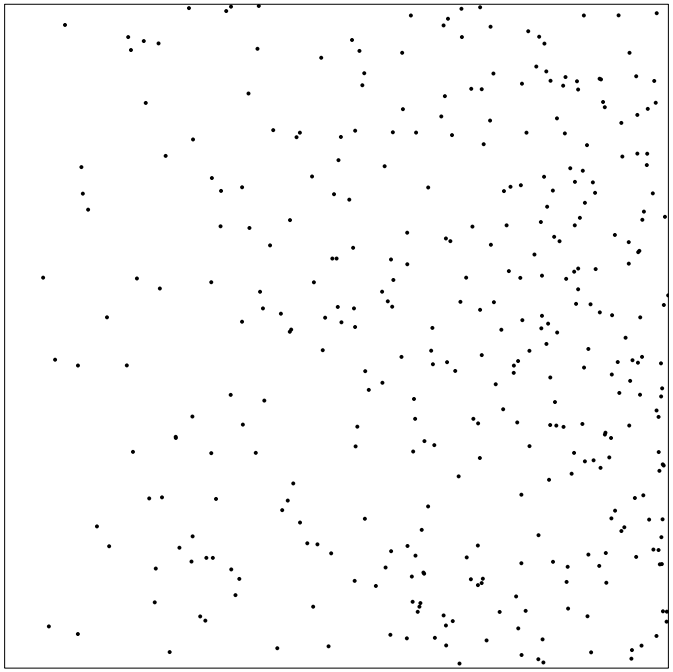
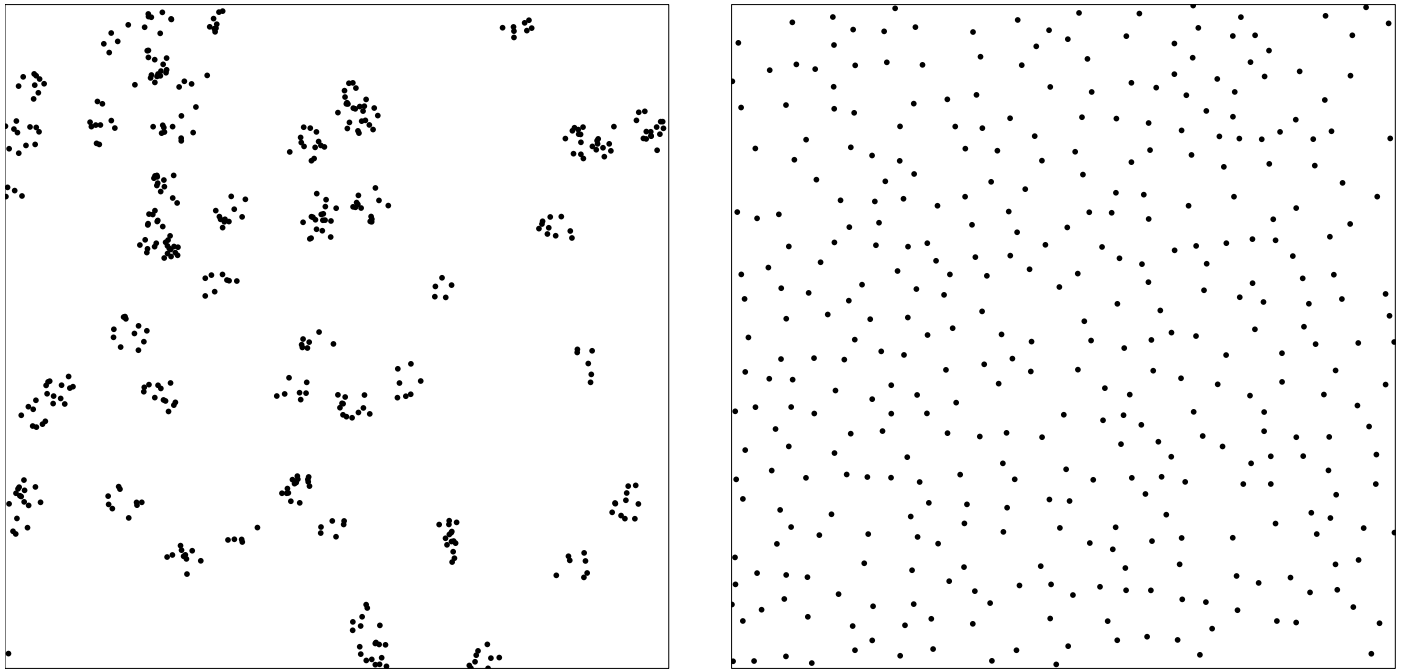


Realization of a general (non-stationary) Poisson process

Poisson-related point processes

- Poisson cluster processes
- Poisson hardcore processes



Realizations of a Poisson cluster process (left) and a Poisson hardcore process (right)

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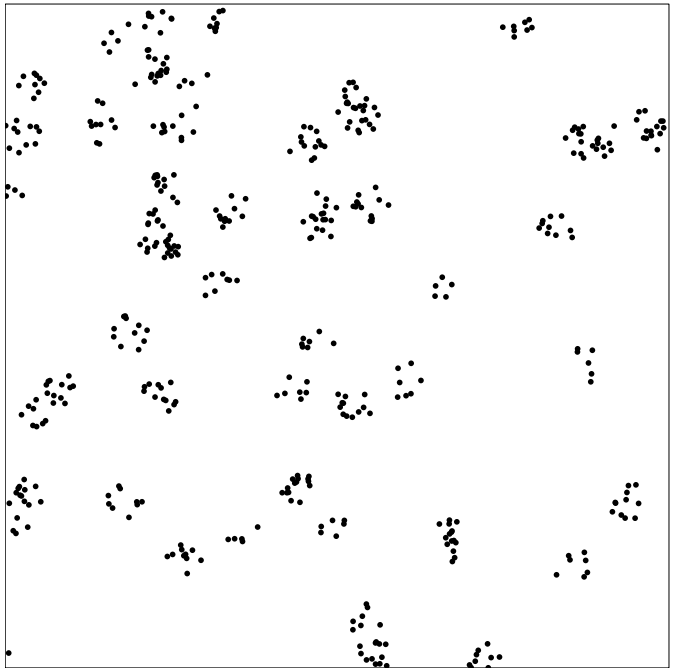

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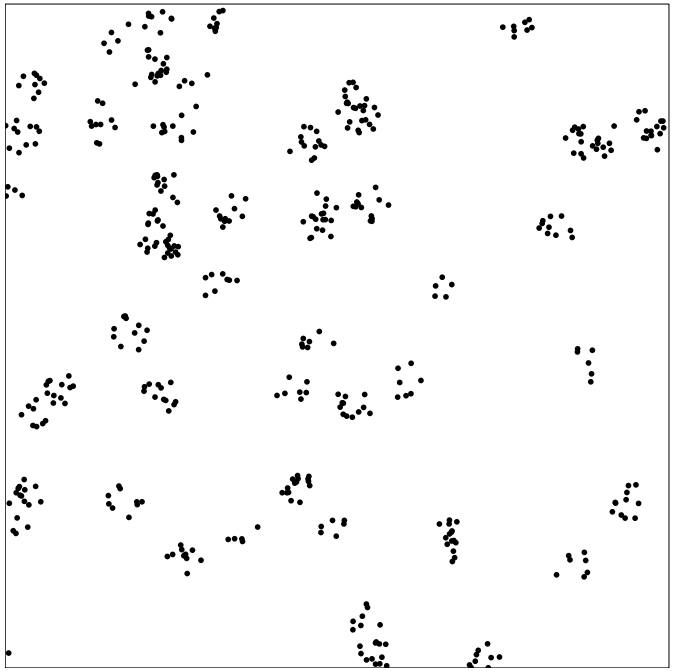

- Three-parametric model with parameters λ_0, λ_1 and R

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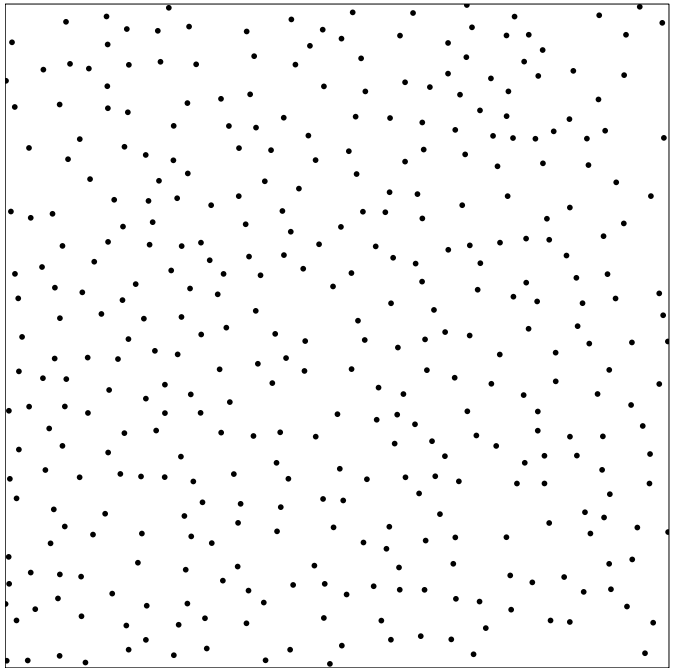

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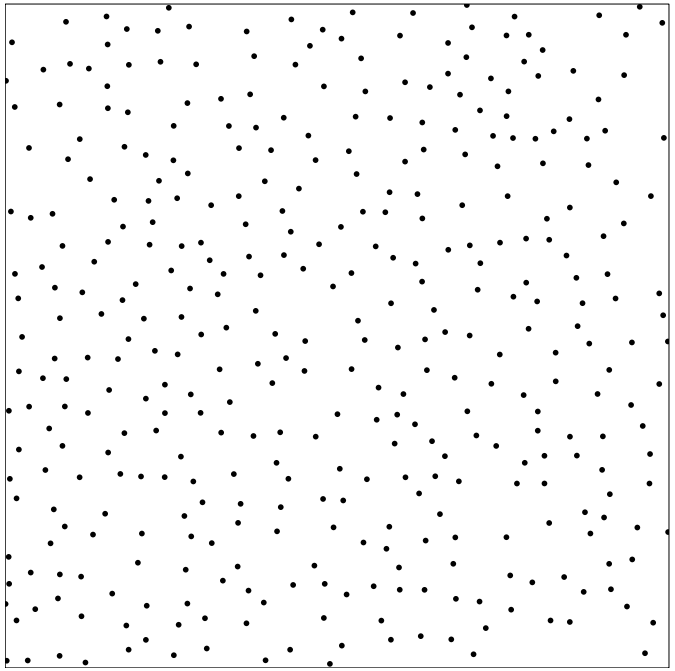

- Two-parametric model with parameters λ and R

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Cox point processes

- A point process X is called a Cox process with random intensity measure Λ if

$$\mathbb{P}(X(B_1) = k_1, \ldots, X(B_n) = k_n) = \mathbb{E}\left(\prod_{i=1}^{n} \frac{\Lambda(B_i)^{k_i}}{k_i!} e^{-\Lambda(B_i)}\right),$$

for all pairwise disjoint, bounded $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^2)$ and $k_1, \ldots, k_n \geq 0$.

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- Conditioning on $\Lambda = \eta$, a Cox process X is a Poisson process with intensity measure η.

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- and $\mathcal{N}_{\mathbb{M}}$ the smallest σ-algebra on $\mathbf{N}_{\mathbb{M}}$ such that $\psi \to \psi(B \times G)$ is measurable for each bounded $B \in \mathcal{B}(\mathbb{R}^2)$ and $G \in \mathcal{B}(\mathbb{M})$.

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Marked point processes

Definition

Let

- $(\Omega, \mathcal{A}, \mathbb{P})$ *some probability space,*
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- *The intensity measure* $\mu : \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(\mathbb{M}) \to [0, \infty]$ *of* X_M *is defined as*

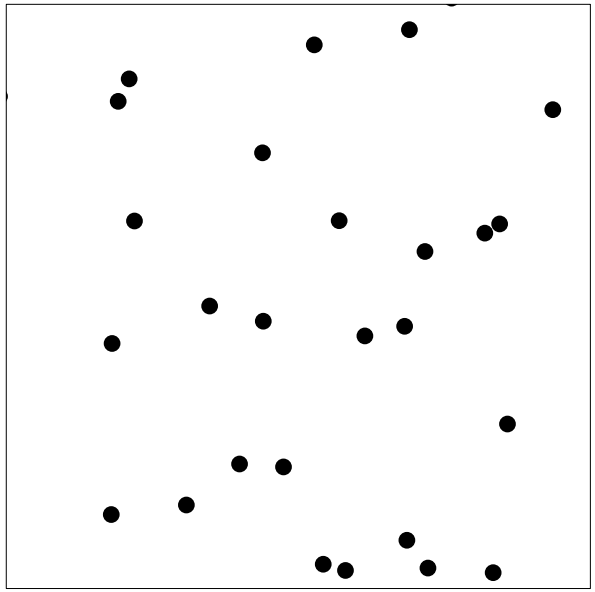
$$\mu(B \times C) = \mathbb{E} X_M(B \times C)\,, \quad B \in \mathcal{B}(\mathbb{R}^2)\,, C \in \mathcal{B}(\mathbb{M})\,.$$

Example: Poisson-Voronoi tessellation

- Let $X = \{X_n\}$ be a stationary Poisson process and consider the Voronoi cell Ξ_n of X_n:

$$\Xi_n = \{x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k| \, \forall k \neq n\}$$

- Then, $\{(X_n, M_n)\}$, where $M_n = \Xi_n - X_n$, is a (stationary) marked point process with mark space $\mathcal{P}^o$



Realization of a Poisson process $\{X_n\}$

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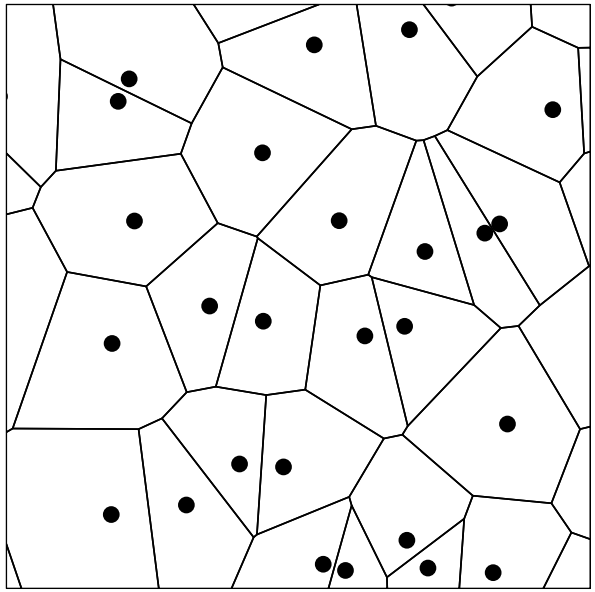
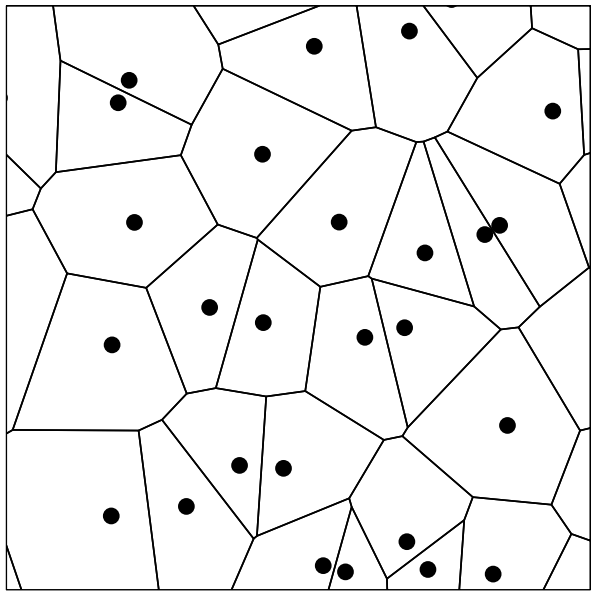


Realization of a Poisson-Voronoi tessellation $\{(X_n, \Xi_n - X_n)\}$

Example: Poisson-Voronoi tessellation

Further (stationary) marked point processes associated with $\{(X_n, \Xi_n - X_n)\}$:

- $\{(X_n, \nu_2(\Xi_n))\}$ with mark space $[0, \infty)$
- $\{(X_n, \nu_1(\partial\Xi_n))\}$ with mark space $[0, \infty)$



Realization of a Poisson-Voronoi tessellation $\{(X_n, \Xi_n - X_n)\}$

Palm mark distribution and Palm distribution

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- *X_M is called stationary if $\{(X_n - x, M_n)\} \stackrel{\mathrm{D}}{=} \{(X_n, M_n)\}$ for each $x \in \mathbb{R}^2$.*

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- *Note that* $P^*_{X_M}(C) = P^o_{X_M}(\mathbf{N}_{\mathbb{M}} \times C)$ *for any* $C \in \mathcal{B}(\mathbb{M})$.

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Definition

Let X_M be a stationary marked point process with Palm mark distribution $\mathbb{P}^*_{X_M}$.

- *A random variable $M^* : \Omega \longrightarrow \mathbb{M}$ distributed according to $\mathbb{P}^*_{X_M}$ is called the typical mark of X_M.*

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- *A random variable $M^* : \Omega \longrightarrow \mathbb{M}$ distributed according to $\mathbb{P}^*_{X_M}$ is called the typical mark of X_M.*
- *If X_M is ergodic, then M^* can be regarded as the mark at a point chosen purely at random out of $\{X_n\}$,i.e.,*

$$\mathbb{E}h(M^*) = \lim_{r\to\infty} \frac{1}{\#\{n : X_n \in [-r,r]^2\}} \sum_{i:\, X_i \in [-r,r]^2} h(M_i)$$

almost surely for each measurable $h : \mathbb{M} \longmapsto [0,\infty)$.

Example: Independent marking

- Let $X = \{X_N\}$ be a point process and
- $M_1, M_2, \cdots : \Omega \to \mathbb{R}$ i.i.d. random variables with some distribution P, which are independent of $\{X_n\}$.

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  - and the typical mark M^* of $X_M = \{(X_n, M_n)\}$ has distribution $P_{X_M}^* = P$.

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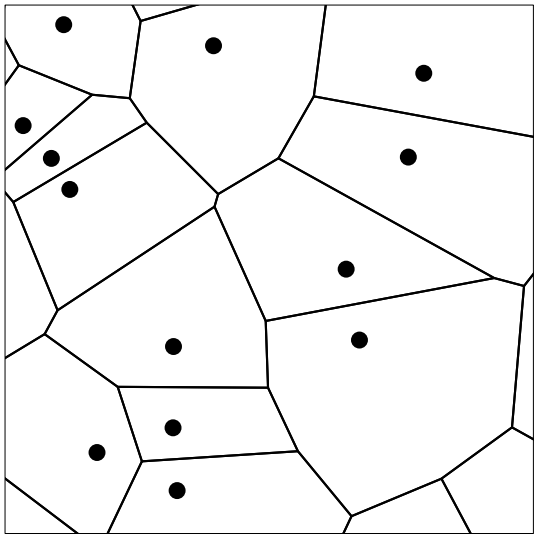

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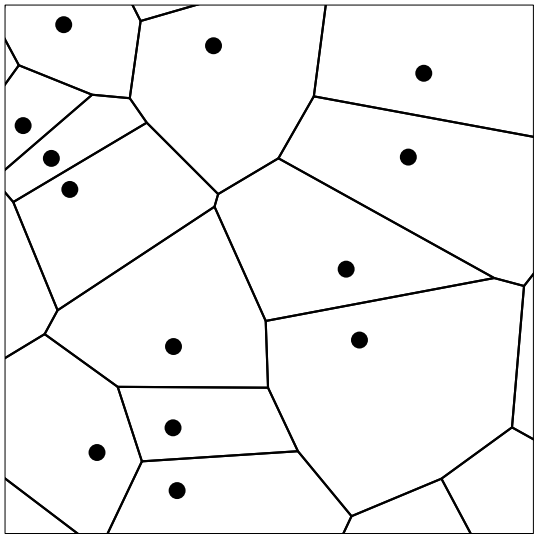

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Palm version of the Poisson-Voronoi tessellation $\{(X_n, \Xi_n - X_n)\}$

- Add the origin $X_0 = o$ to the stationary Poisson process $X = \{X_n\}$.

Example: Poisson-Voronoi tessellation

- Then, by Slivnyak's theorem, the point process $X^o = \{X_n^o\}$, where $\{X_n^o\} = \{X_0, X_1, X_2, \ldots\}$, is a Palm version of $X = \{X_1, X_2, \ldots\}$.

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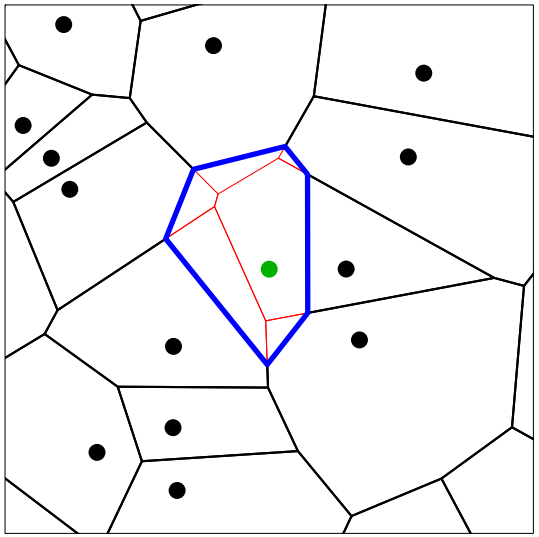

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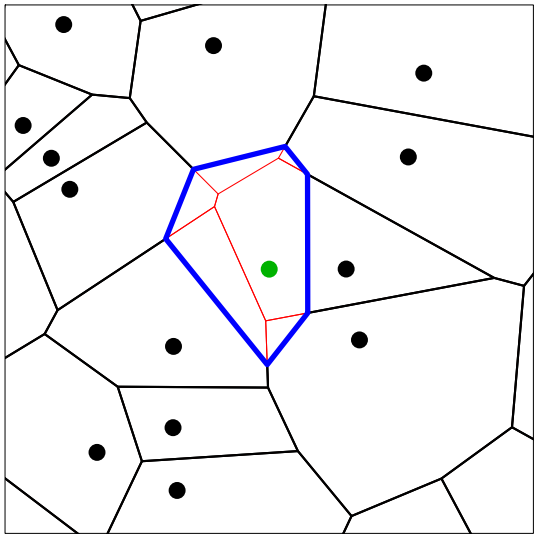

Realization of the Palm version $\{(X_n^o, \Xi_n^o - X_n^o)\}$

- The typical mark Ξ^* of X_M is given by $\Xi^* = \Xi_0^o$

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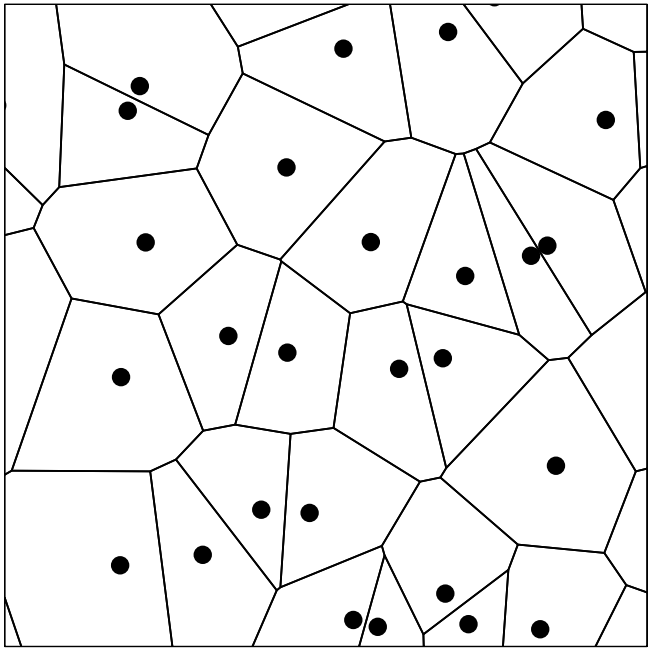

Realization of a PVT $\{(X_n, \Xi_n - X_n)\}$

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- The Laguerre cell Ξ_n of X_n is given by

$$\Xi_n = \{x \in \mathbb{R}^2 : |x - X_n|^2 - R_n^2 \leq |x - X_k|^2 - R_k^2, \ \forall k \neq n\}$$

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$$T = \{(X_n, \Xi_n - X_n) \text{ such that } \mathrm{int}(\Xi_n) \neq \emptyset\}$$

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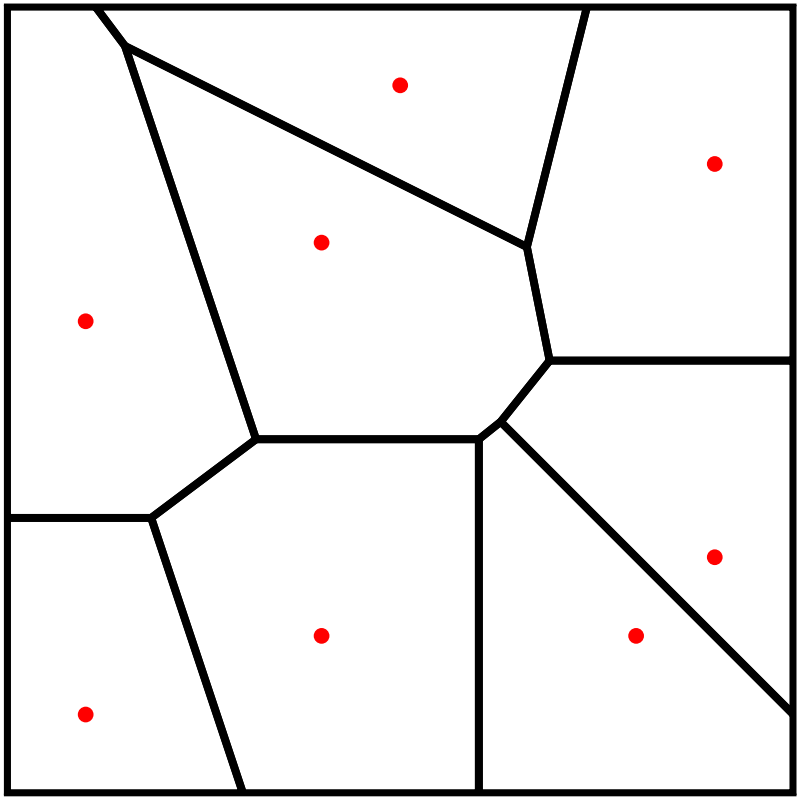
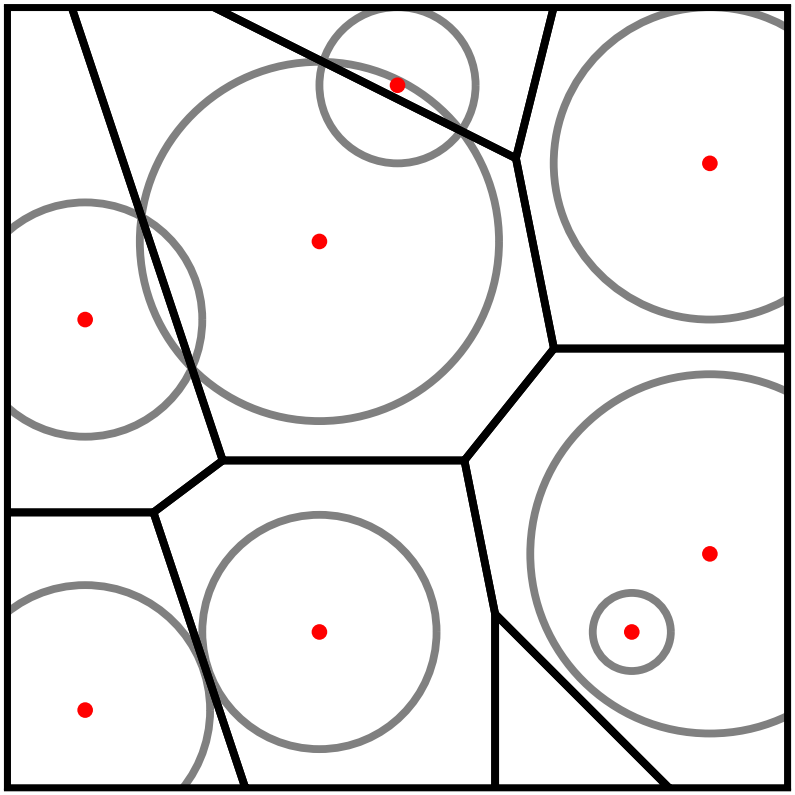
Poisson-Laguerre tessellation

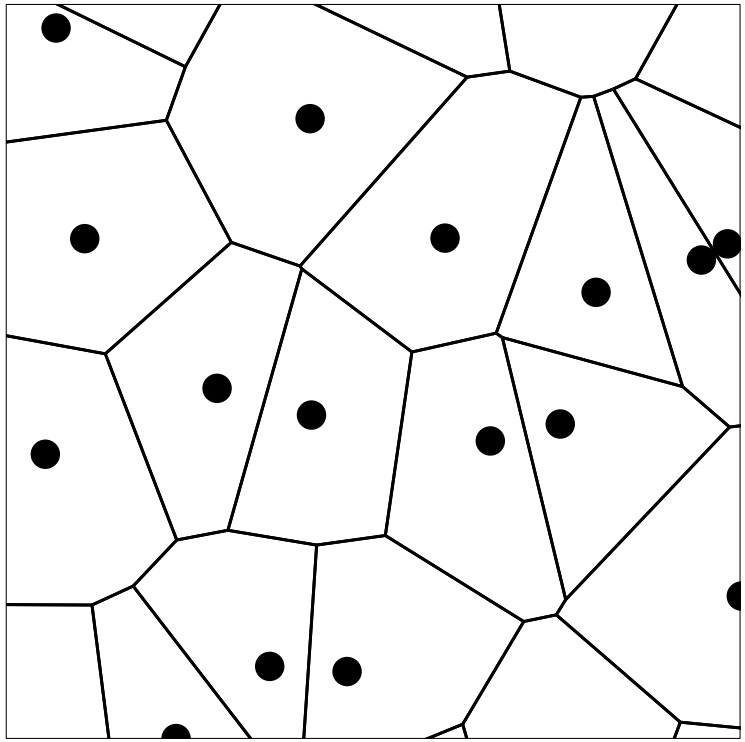




Cutout of Voronoi tessellation (left) and cutout of Laguerre tessellation on the the same set of seed points (right)

Poisson-Delaunay tessellation

- Consider a Voronoi tessellation $T = \{(X_n, \Xi_n - X_n)\}$ induced by a stationary Poisson process $\{X_n\}$



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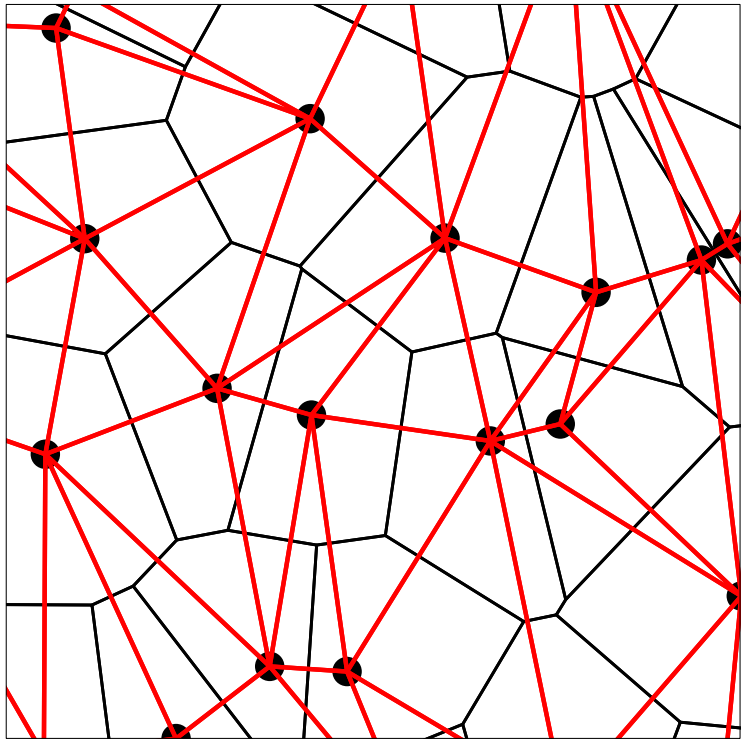

- For each vertex X'_n of T construct the cell Ξ'_n as the triangle formed by the nuclei $X_{i_1}, X_{i_2}, X_{i_3}$ of the three neighboring Voronoi cells.

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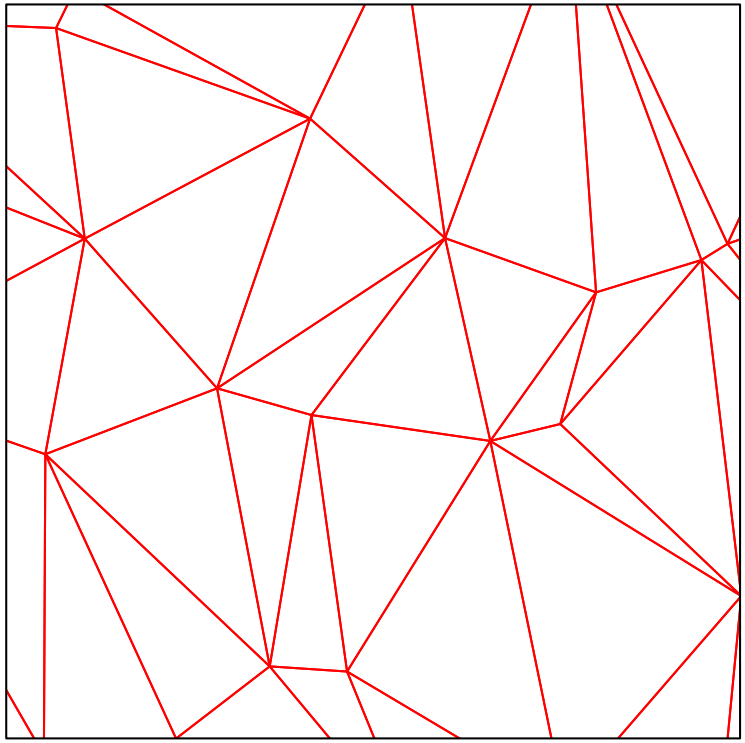

- For each vertex X'_n of T construct the cell Ξ'_n as the triangle formed by the nuclei $X_{i_1}, X_{i_2}, X_{i_3}$ of the three neighboring Voronoi cells.
- Then $T' = \{(X'_n, \Xi'_n - X'_n)\}$ is called a Poisson-Delaunay tessellation.

Poisson line tessellation

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- $\ell(\Phi_n, R_n) = \{(x, y) \in \mathbb{R}^2 : x \sin \Phi_n - y \cos \Phi_n = R_n\}$ the line with direction Φ_n and signed distance R_n to the origin $o \in \mathbb{R}^2$

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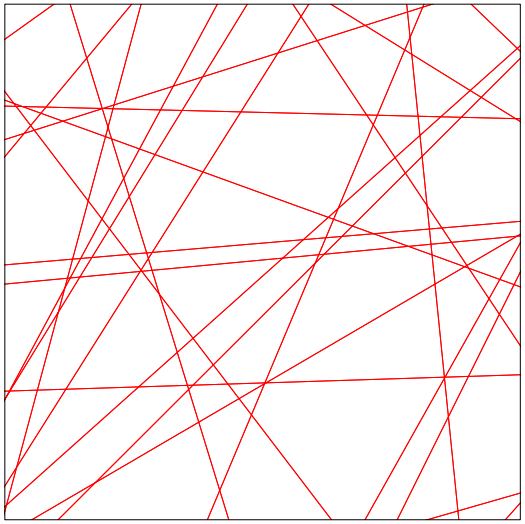
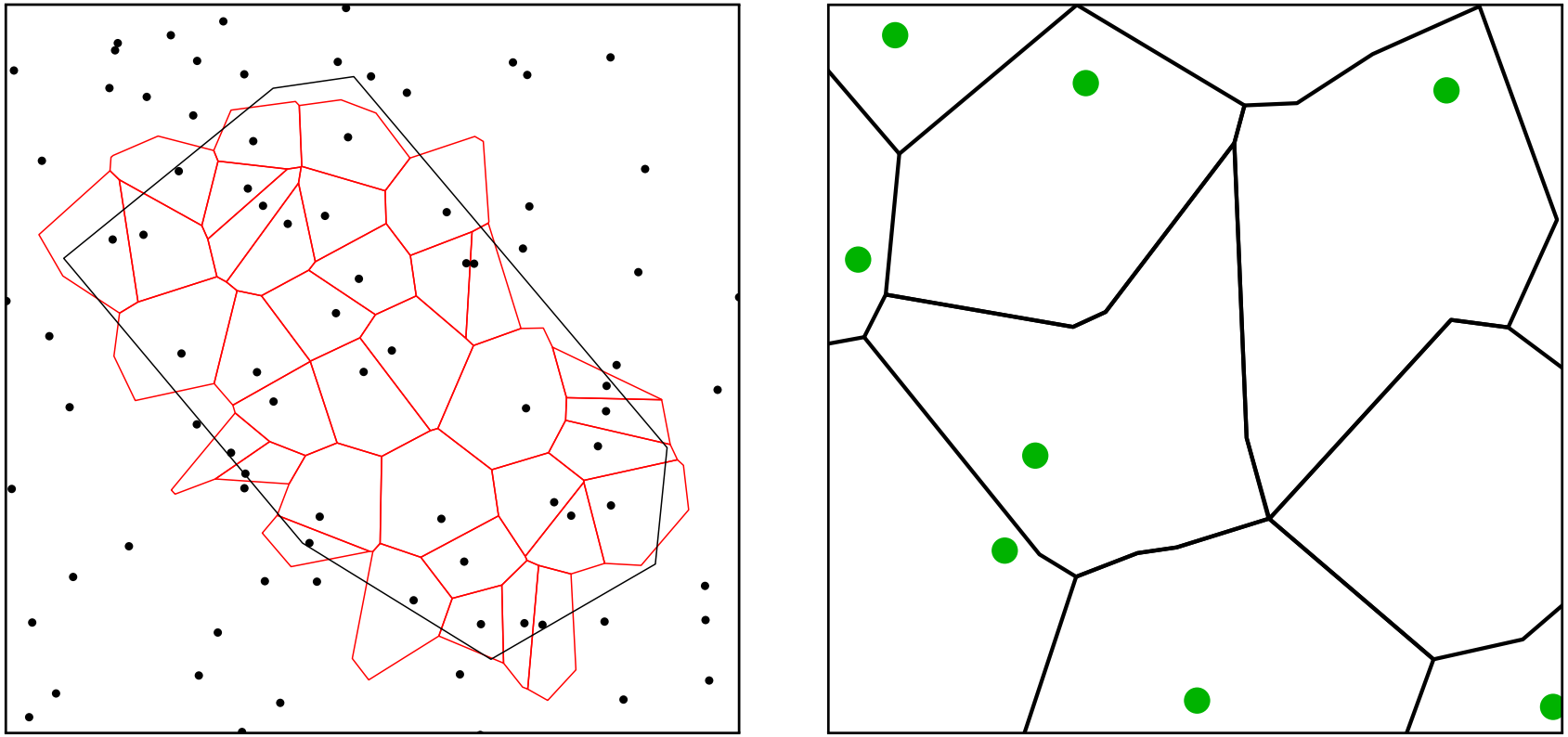
Then, $\{\ell(\Phi_n, R_n)\}$ is called a Poisson line process, where $T^{(1)} = \bigcup_{n \in \mathbb{Z}} \ell(\Phi_n, R_n)$ is the edge set of a Poisson line tessellation (PLT).



Realization of a Poisson line tessellation

Tessellations with general (not necessarily convex) cells

- Aggregate Voronoi tessellations



Construction principle (left) and cutout of an aggregate tessellation (right)

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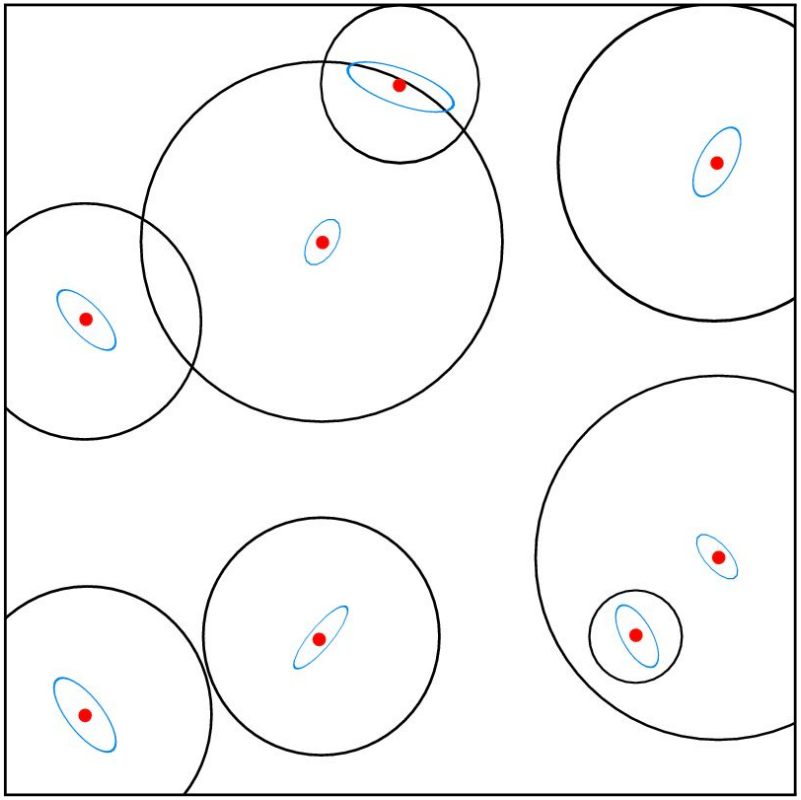
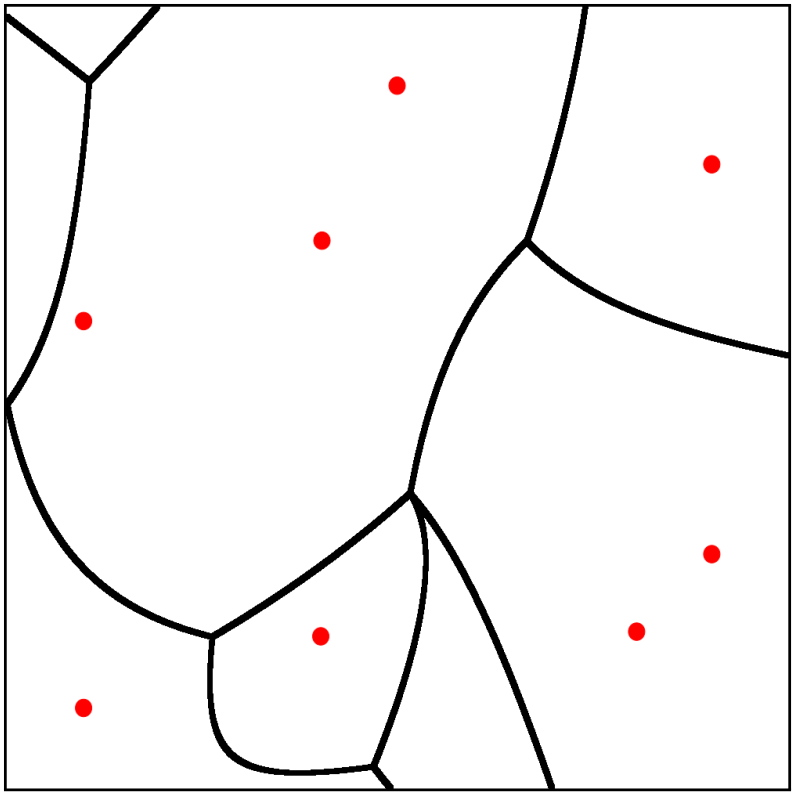
Generalized Laguerre tessellations





Seed points X_n, radii R_n, ellipse-representation of matrices A_n (left), and cutout of generalized Laguerre tessellation (right)

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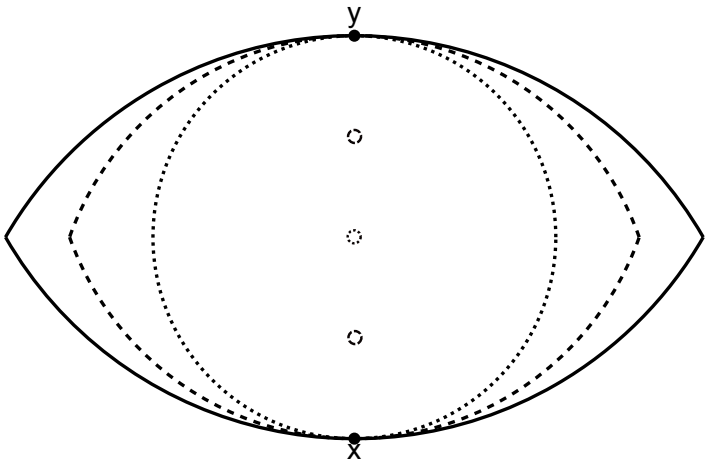

Illustration of the intersection $A_\beta(x,y)$ of the two balls:

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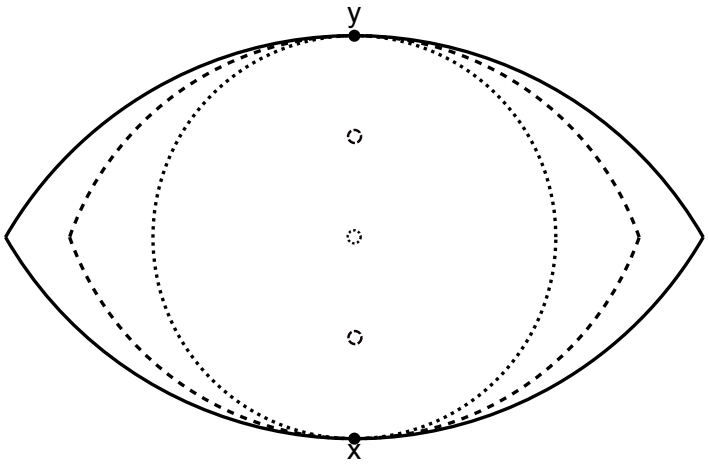

Illustration of the intersection $A_\beta(x, y)$ of the two balls: for $\beta = 1$ (dotted), $\beta = 1.5$ (dashed) and $\beta = 2$ (solid)

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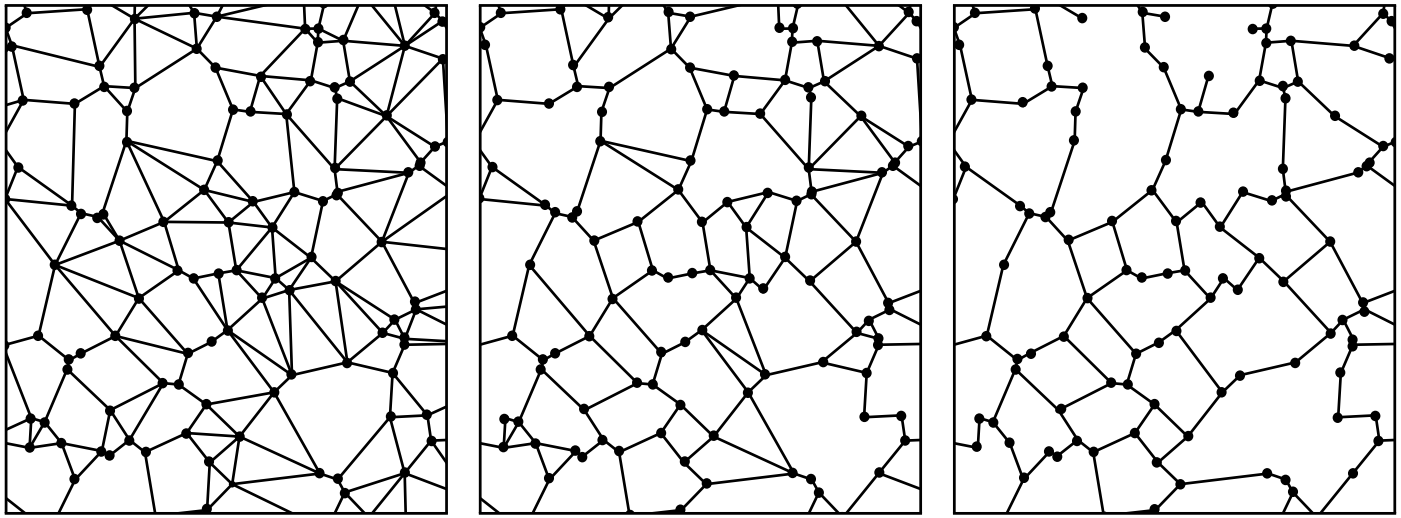

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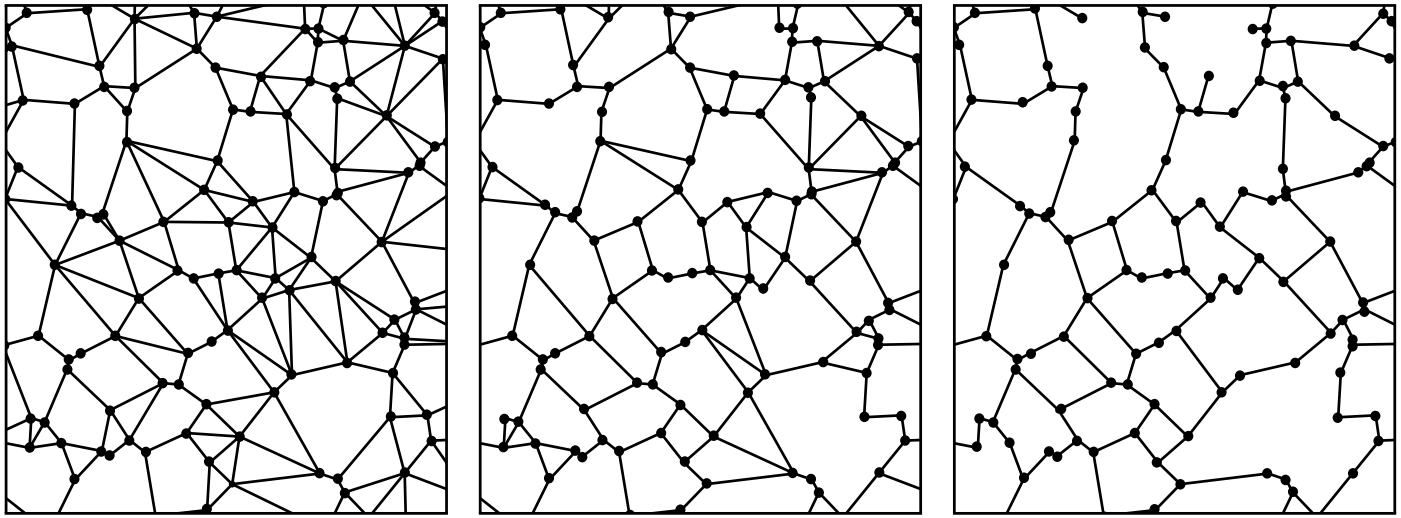

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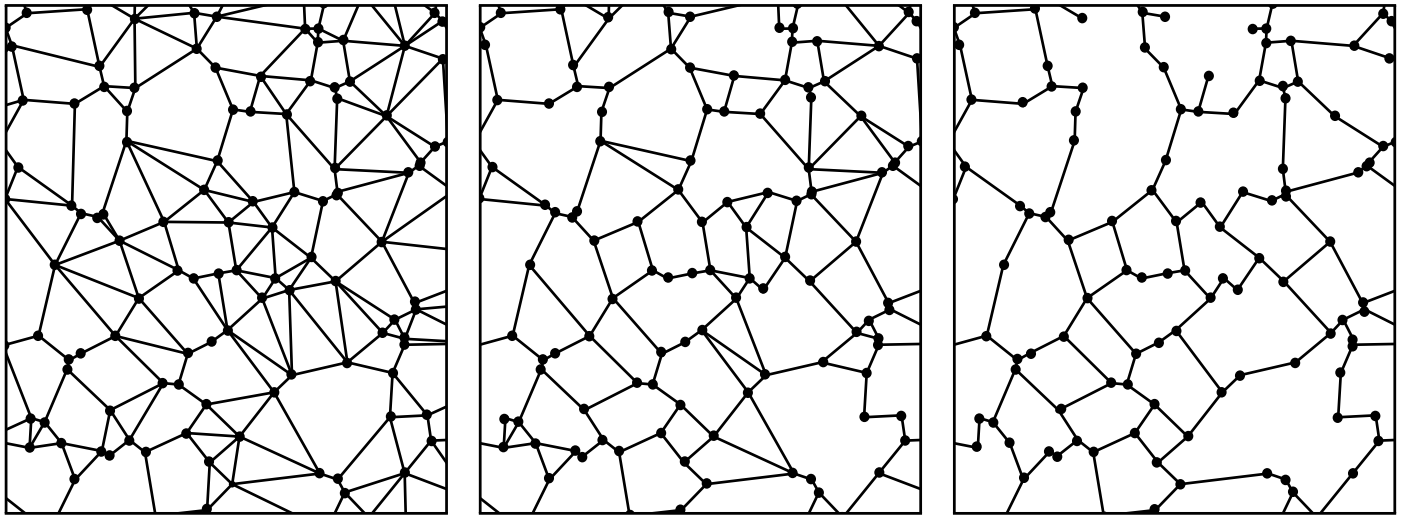

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General idea

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- Use a suitable stopping rule to reduce runtime.

Radial simulation of Poisson processes

Theorem

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- $\lambda > 0$ *be an arbitrary, but fixed number,*
- $Y_1, Y_2, \ldots$ *i.i.d.* $Exp(1)$*–distributed,*
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Proof Idea: Show that $X(B) \sim Poi(\lambda\nu_2(B))$ and $X(B_1), \ldots, X(B_n)$ are independent for $B_1, \ldots, B_n \in \mathcal{B}$ with $B_i \cap B_j = \emptyset$ for $i \neq j$, e.g.,

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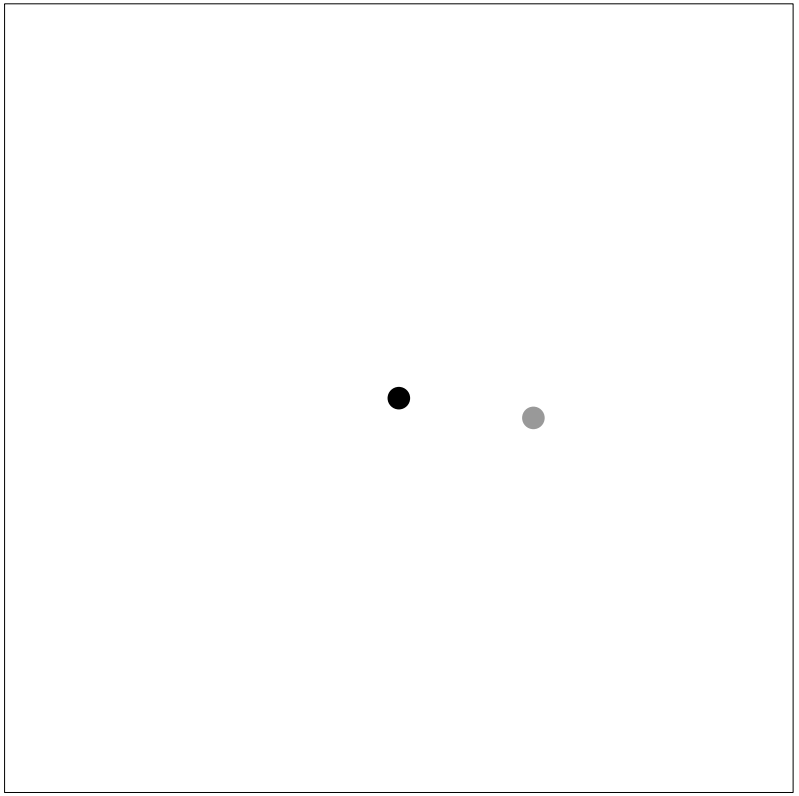
Then $\{X_n\}$ *is a stationary Poisson process in* $\mathbb{R}^2$ *with intensity* λ.

Proof Idea: Show that $X(B) \sim Poi(\lambda\nu_2(B))$ and $X(B_1), \ldots, X(B_n)$ are independent for $B_1, \ldots, B_n \in \mathcal{B}$ with $B_i \cap B_j = \emptyset$ for $i \neq j$, e.g., $X(B(o, r)) = \#\{n : R_n \leq r\} = \#\{n : \sum_{k=1}^{n} Y_k \leq \lambda\pi r^2\} \sim Poi(\lambda\pi r^2)$.

Radial simulation of Poisson processes

- **Algorithm:**
  - Simulate $Y_n \sim Exp(1)$, $U_n \sim U[0, 2\pi)$ independent of $Y_1, \ldots, Y_{n-1}, U_1, \ldots, U_{n-1}$
  - Construct $X_n = (R_n \cos U_n, R_n \sin U_n)$ with $R_n = \sqrt{\sum_{k=1}^{n} Y_k/(\pi\lambda)}$



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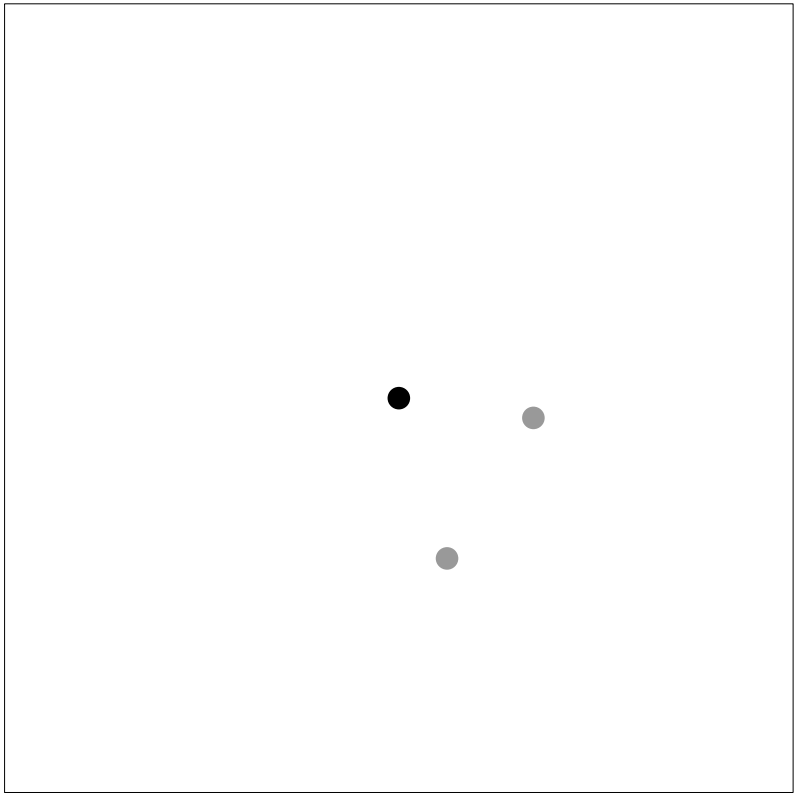

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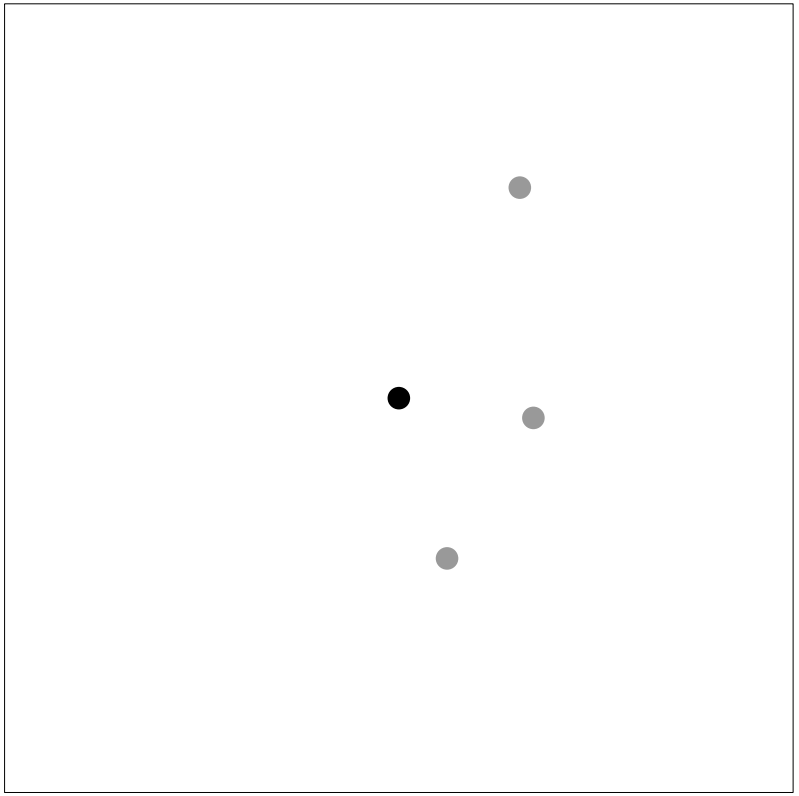

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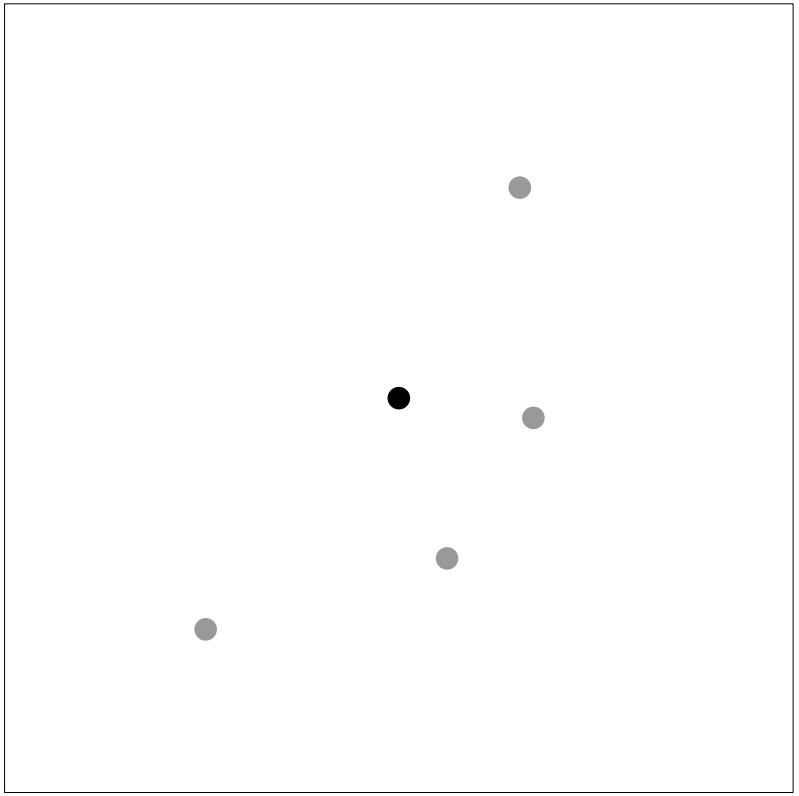

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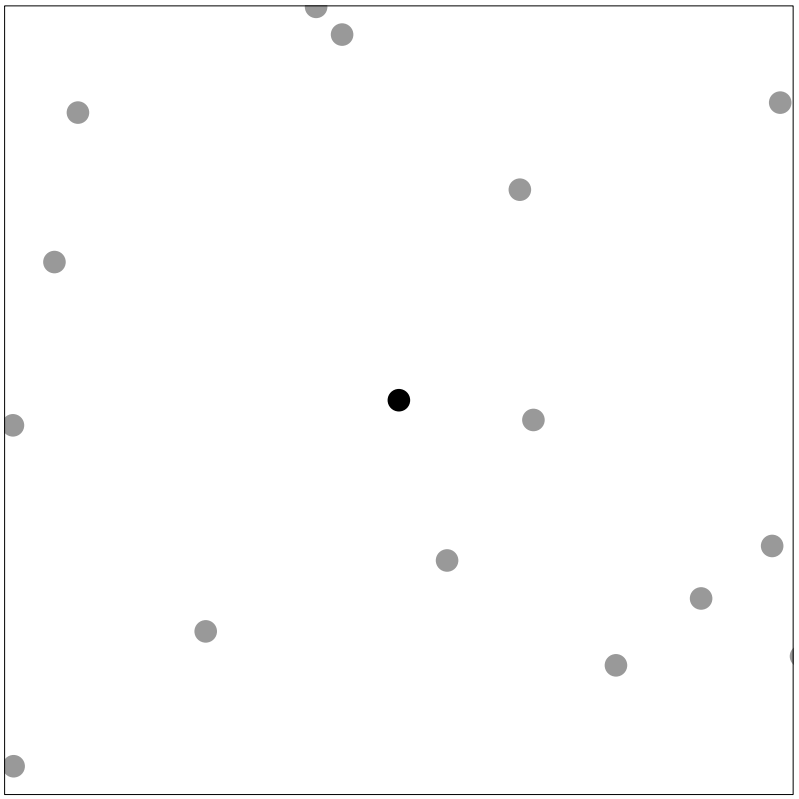

- Stop if $R_n > a/\sqrt{2}$, where a is the side length of the sampling window

Slivnyak's theorem

Theorem

Let X be a stationary Poisson process with some intensity $\lambda > 0$. Then

$$\mathbb{P}(X^0 \in A) = \mathbb{P}(X \cup \{o\} \in A)\,,$$

where X^0 is the Palm version of X, i.e., X^0 is distributed according to the Palm distribution P_X^0 of X.

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Proof Consider void probabilities $\mathbb{P}(X^0(C) = 0)$, $C \subset \mathbb{R}^2$ compact. Then $\mathbb{P}(X^0(\{o\}) = 1) = \mathbb{P}(X(\{o\}) = 0) = 1$ by definition. Furthermore, if $o \notin C$, then

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&= \lim_{\varepsilon \searrow 0} \frac{\mathbb{P}(X(C) = 0)\mathbb{P}(X(B(o,\varepsilon)\backslash C) = 1)}{\mathbb{P}(X(B(o,\varepsilon)) = 1} \\
&= \mathbb{P}(X(C) = 0)\,.
\end{aligned}$$

The typical Voronoi cell

- Let
  - $\{X_n\}$ be a stationary Poisson process and $T = \{\Xi_n\}$ the induced Poisson-Voronoi tessellation (PVT), i.e.,

$$\begin{aligned} \Xi_n &= \{x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k| \, \forall k \neq n\} \\ &= \bigcap_{k \in \mathbb{N}: k \neq n} H(X_n, X_k) \end{aligned}$$

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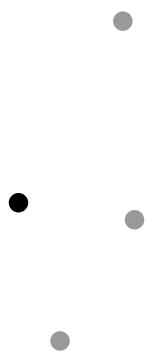
- Slivnyak's theorem yields

$$\Xi^* = \bigcap_{n=1}^{\infty} H(o, X_n)$$

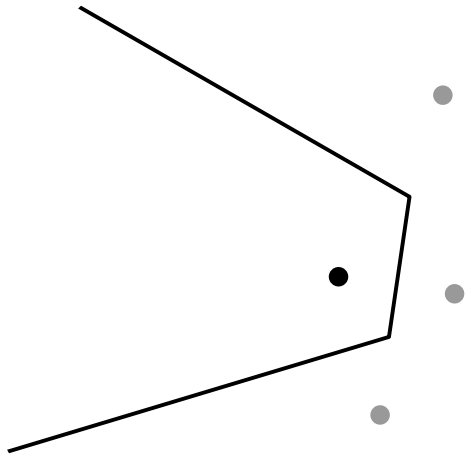
Local simulation of the typical cell of PVT

- **Algorithm:**
  - Place point at o Simulate points X_1, X_2, X_3 of Poisson process X radially



Local simulation of the typical cell of PVT

- **Algorithm:**
  - Intersect halfplanes $H(o, X_1)$, $H(o, X_2)$ and $H(o, X_3)$



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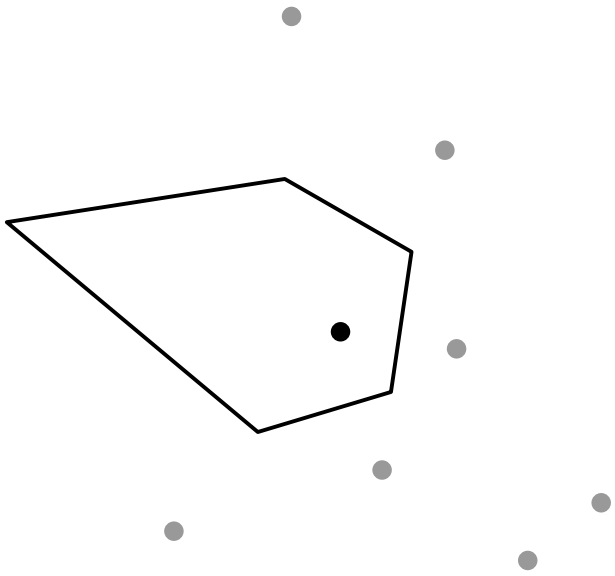
- **Algorithm:**
  - Simulate further points of X and intersect halfplanes $\Rightarrow$ inital cell



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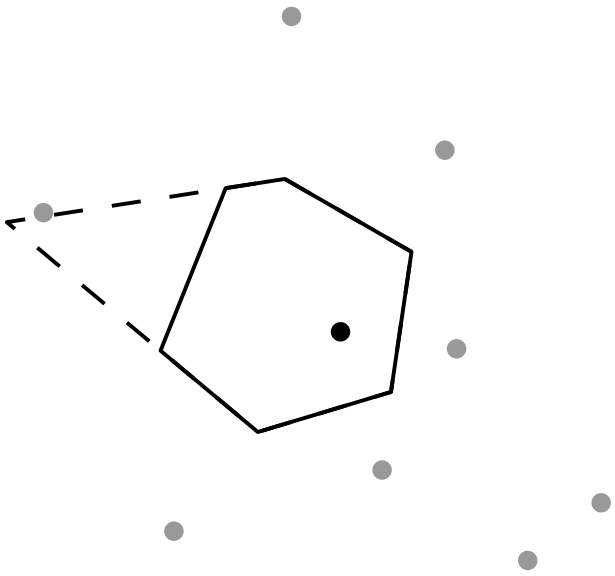
- **Algorithm:**
  - Simulate further points of X and intersect initial cell with halfplanes



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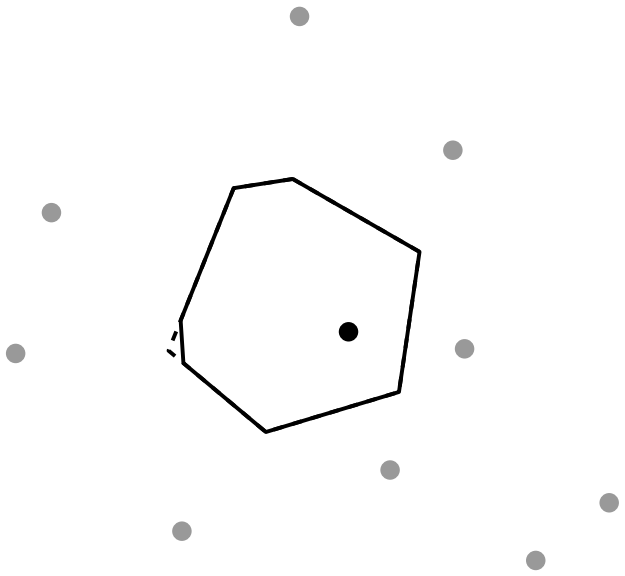
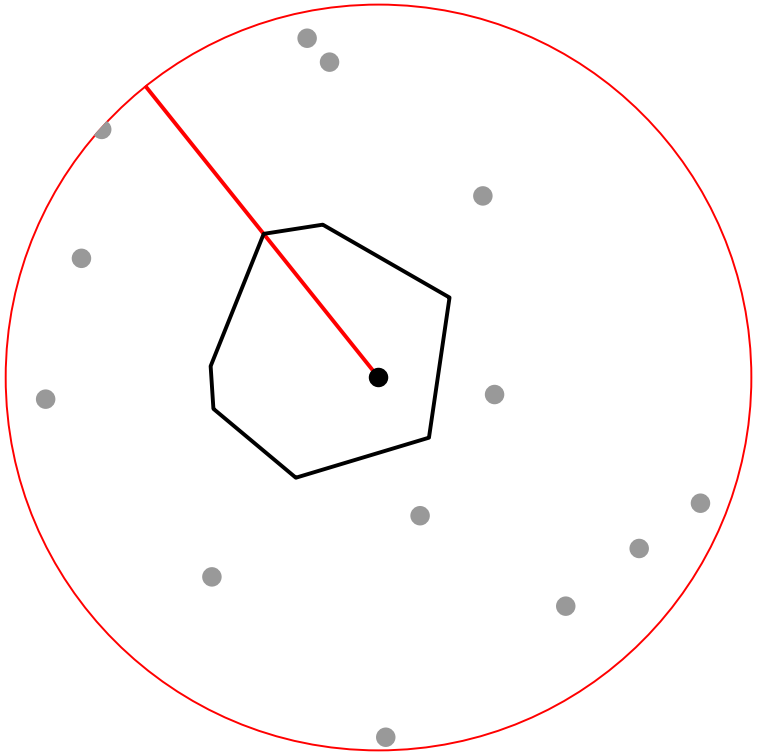


Local simulation of the typical cell of PVT

- **Algorithm:**
  - Stop if $|X_n| > 2\max_{i=1,\dots,m}|V_m|$, where $V_1, \dots, V_m$ are the vertices of the current modification of the initial cell



Contents

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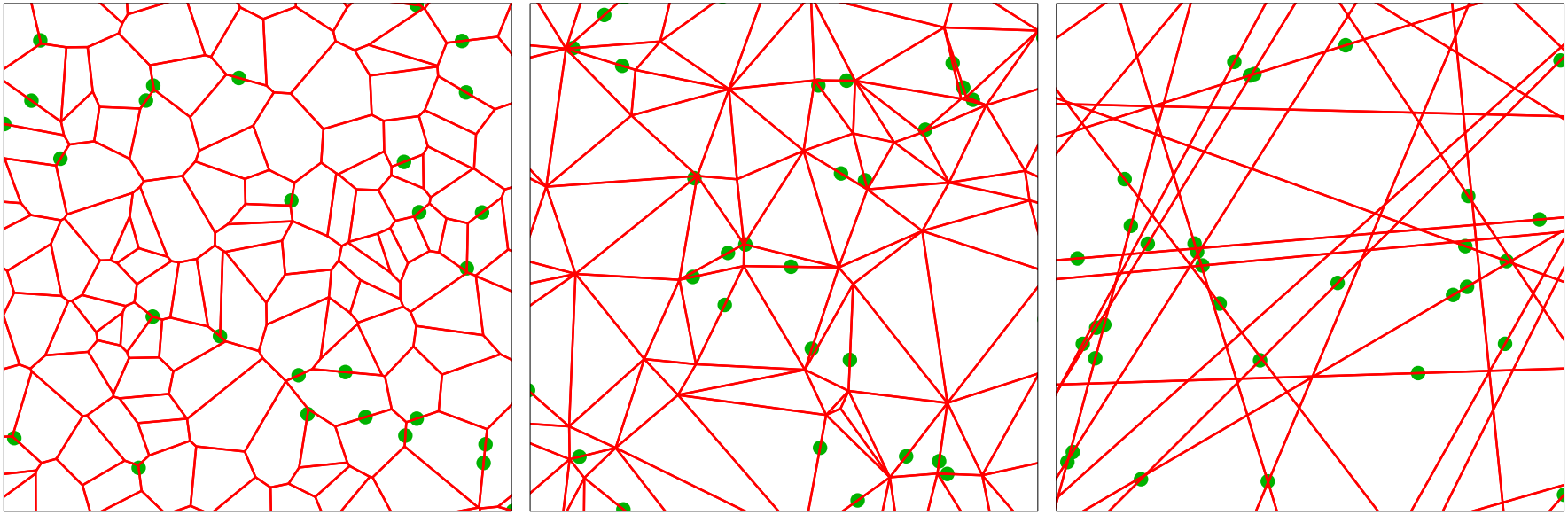
Cox processes on random tessellations

Examples



Realizations of Cox processes on the edge sets
of various random tessellations

Local simulation of typical Cox-Voronoi cells

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Slivnyak's theorem for Cox processes

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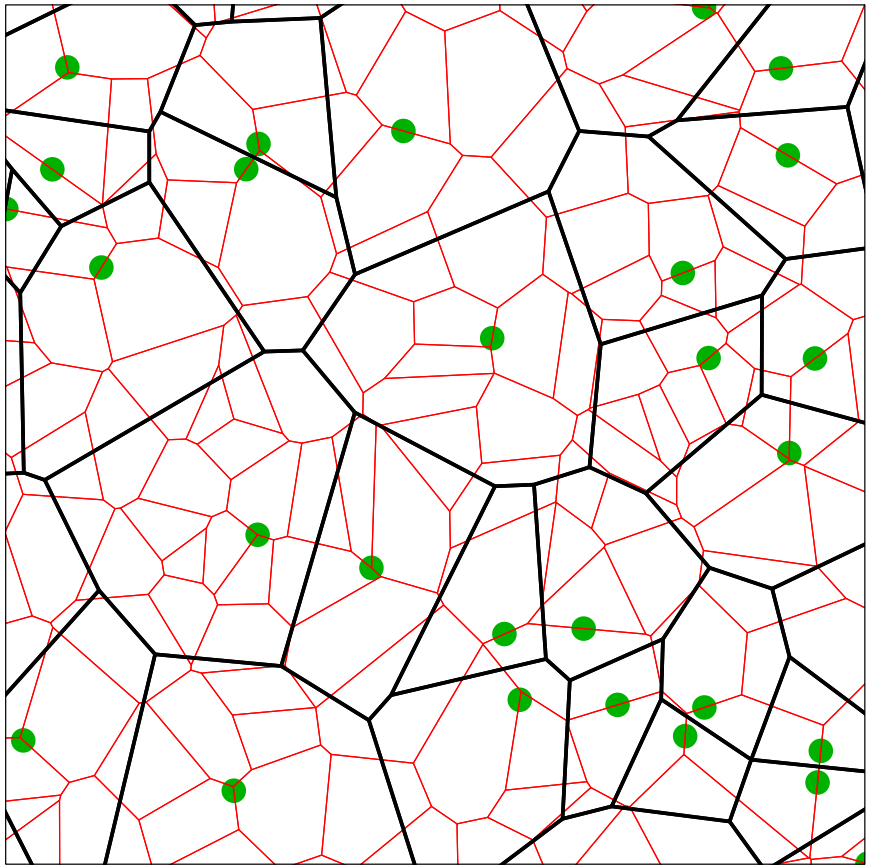
- Thus, Λ^0 is given by $\Lambda^0(\,\cdot\,) = \lambda_\ell \nu_1(\,\cdot\, \cap \widetilde{T}^{(1)})$, where $\widetilde{T}$ can be regarded as conditional version of T under the condition that $o \in T^{(1)}$.

Cox processes on Poisson-Voronoi tessellations



Cox process on PVT and its Voronoi tessellation

Representation of $\widetilde{T}$ for Poisson-Voronoi tessellations

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- R^2 *gamma distributed with parameters* 1.5 *(shape) and* $1/(\lambda\pi)$ *(scale),*
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Then $\widetilde{T}$ is the Voronoi tessellation induced by $\{X_n\}_{n=1}^{\infty}$, where

- *X_1 and X_2 are given by the points $X_1 = (\sqrt{R^2 - \widetilde{R}^2 R^2}, \widetilde{R}R)$ and $X_2 = (\sqrt{R^2 - \widetilde{R}^2 R^2}, -\widetilde{R}R)$, respectively, rotated around o with angle Φ,*
- *$\{X_n\}_{n=3}^{\infty}$ is distributed according to a stationary Poisson process in $\mathbb{R}^2 \backslash B(o, r)$ with intensity λ given $R = r$.*

Representation of $\widetilde{T}$ for Poisson-Voronoi tessellations

Theorem

Let T be a PVT with intensity $\gamma = 2\sqrt{\lambda}$ induced by a stationary Poisson process with intensity λ. Let $R^2, \widetilde{R}^2$ and Φ be independent random variables, where

- R^2 *gamma distributed with parameters* 1.5 *(shape) and* $1/(\lambda\pi)$ *(scale),*
- $\widetilde{R}^2$ *beta distributed with parameters* 1 *and* $1/2$*,*
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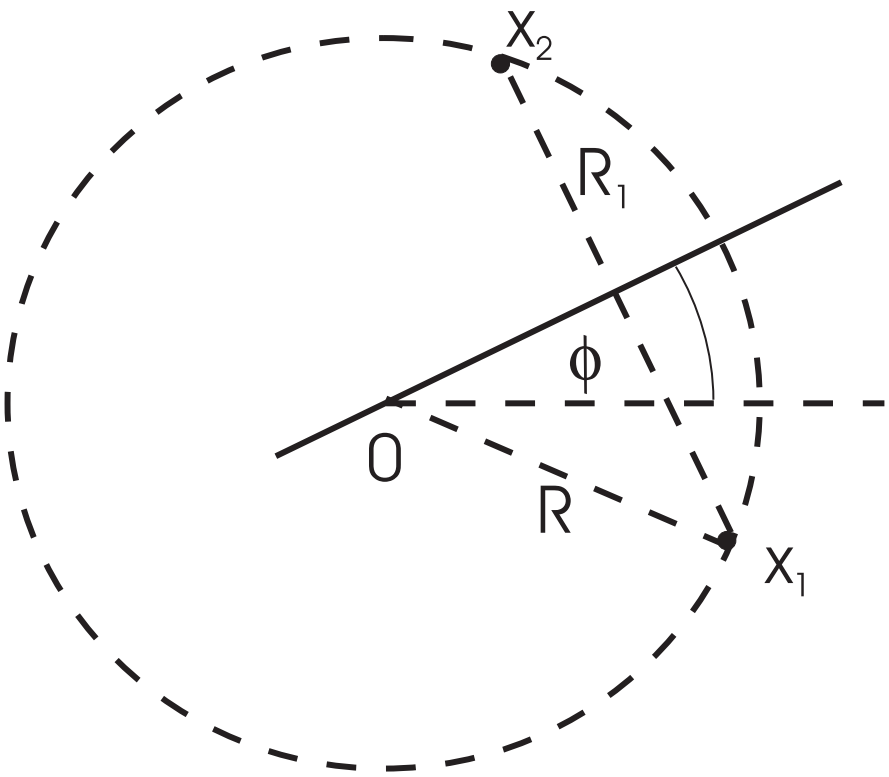
Proof See Baumstark & Last (2007)

Typical Voronoi cell of Cox processes on PVT



Line segment through the origin with the generating points X_1 and X_2, where $R_1 = R\widetilde{R}$

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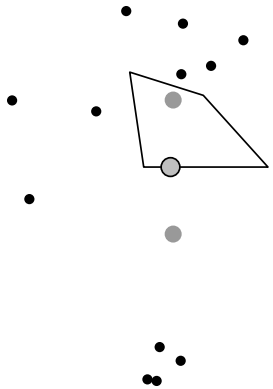

- Simulate two points X_1 and X_2 (grey) generating the segment through o,

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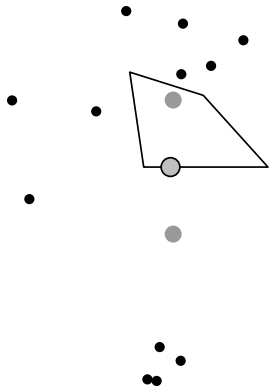

- Simulate two points X_1 and X_2 (grey) generating the segment through o,
- Simulate points $X_3, X_4, \ldots$ of a stationary Poisson process in $\mathbb{R}^2 \backslash B(o, r)$ with intensity λ given $R = r$.

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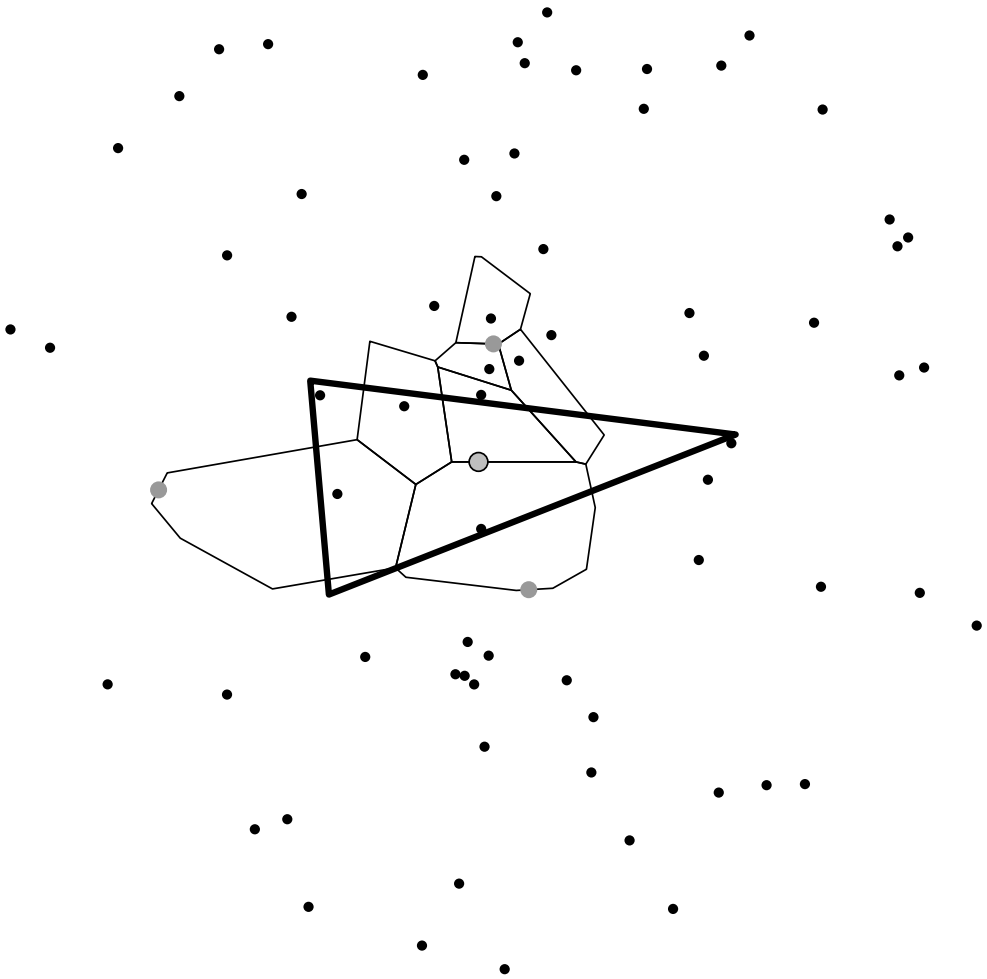
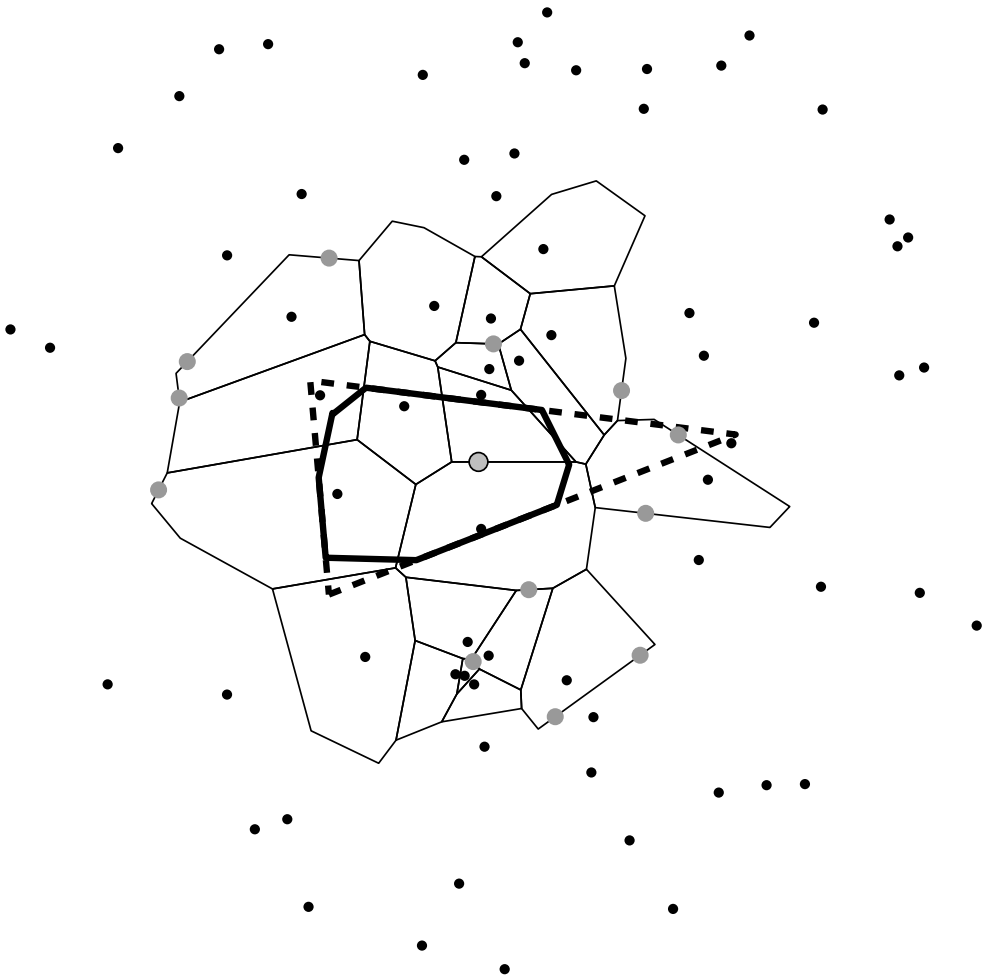


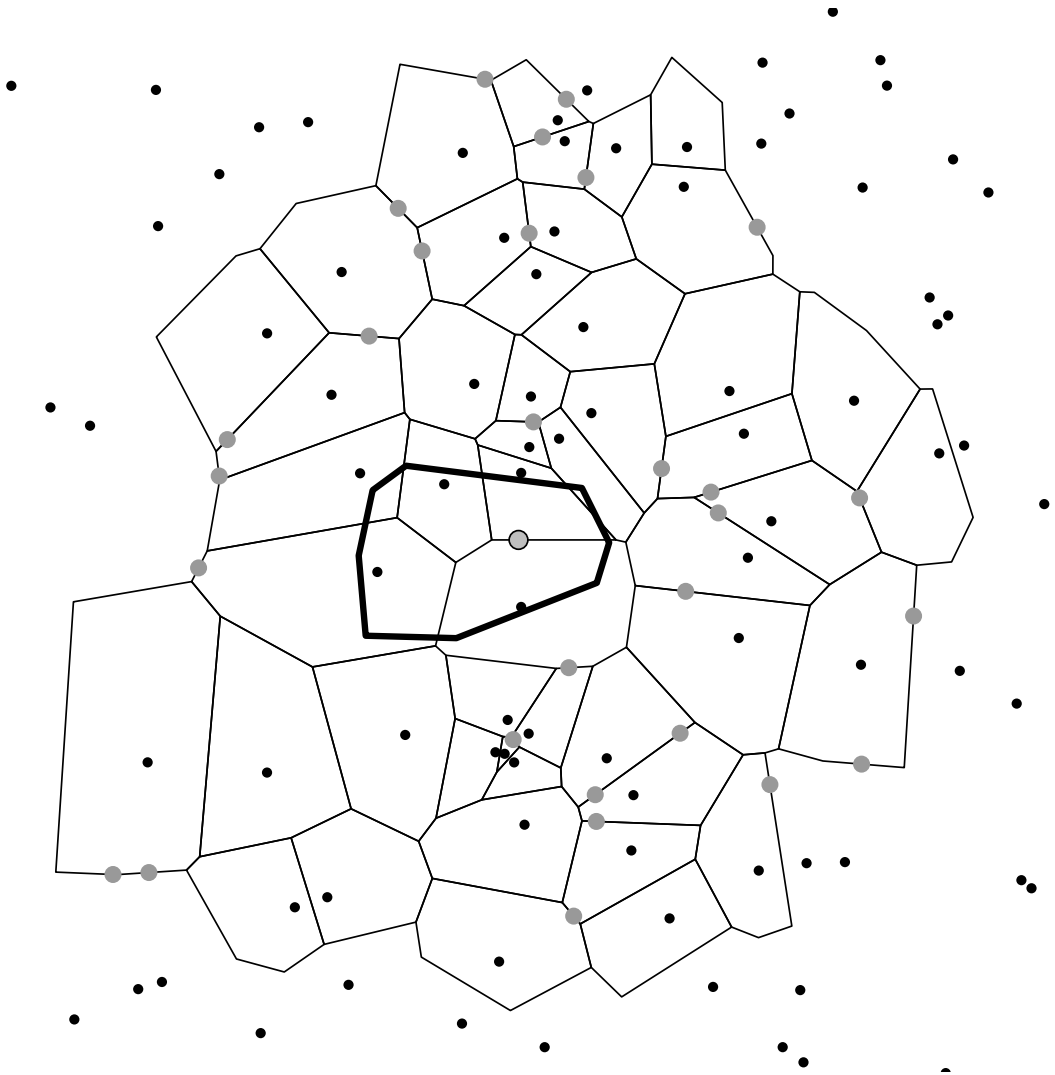
Place points on the edges of underlying Voronoi cells and construct Initial cell

Typical Voronoi cell of Cox processes on PVT



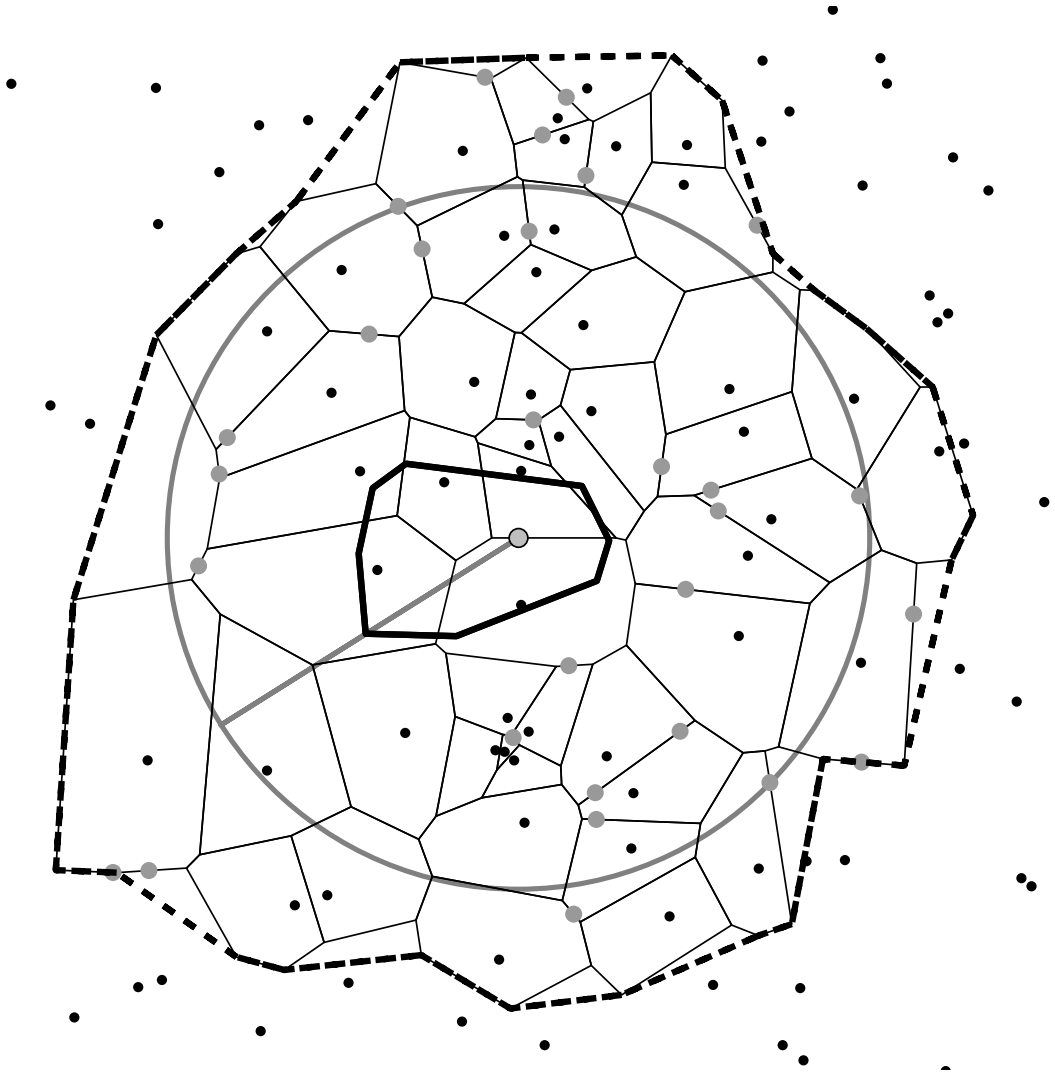
Intersect initial cell by bisectors

Typical Voronoi cell of Cox processes on PVT



Stop

Typical Voronoi cell of Cox processes on PVT



Stopping criterion

General representation of $\widetilde{T}$ for stationary tessellations

Theorem

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Then, for any measurable function $h : \mathbf{N}_{\mathcal{F}} \to [0, \infty)$,

$$\mathbb{E}h(\widetilde{T}) = \frac{1}{\mathbb{E}\,\nu_1(E^*)}\mathbb{E}\left(\nu_1(E^*)\,h(T^* - Z)\right),$$

where the random variable Z is uniformly distributed on E^* given T^*.

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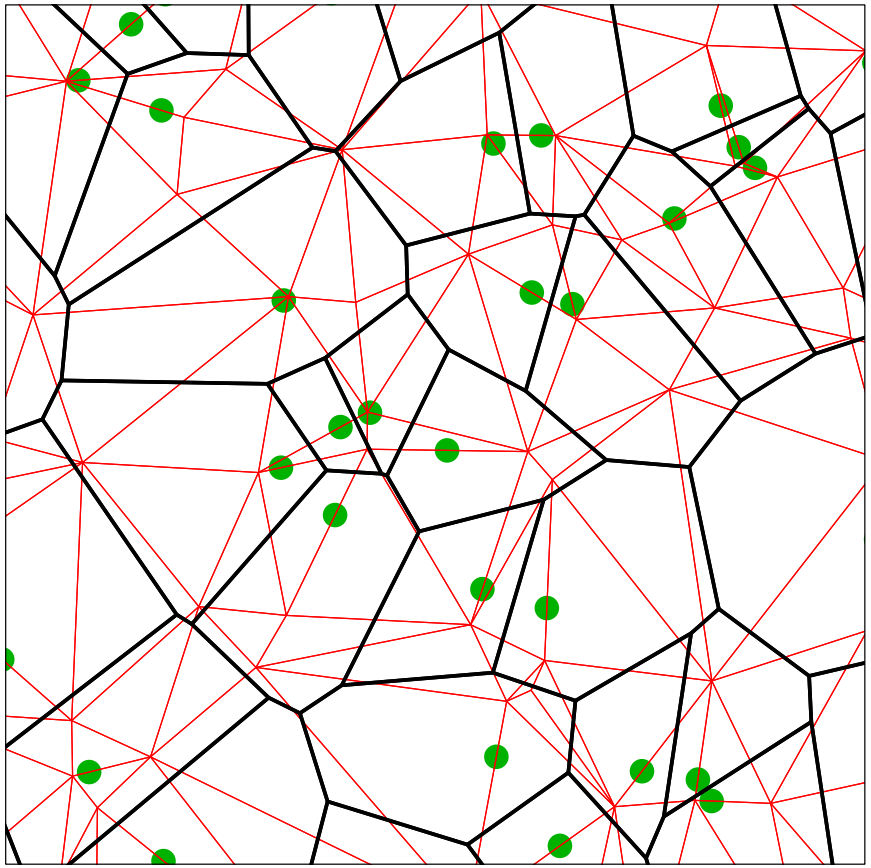
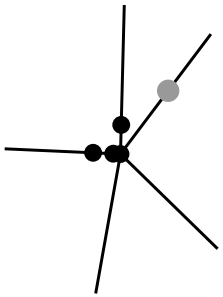
- The distribution of $\widetilde{T}$ can be expressed by the distribution of (T^*, E^*).
- If T is a PDT, then the vertices of T form a stationary Poisson process
- and (T^*, E^*) can be easily simulated using Slivnyak's theorem.

Cox processes on Poisson-Delaunay tessellations



Cox process on PDT and its Voronoi tessellation

Typical Voronoi cell of Cox processes on PDT



Start: Simulate typical edge star E^* using Slivnyak's theorem

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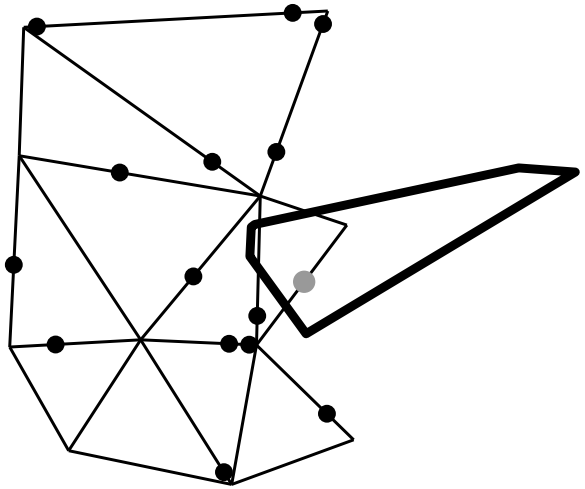

Initial cell

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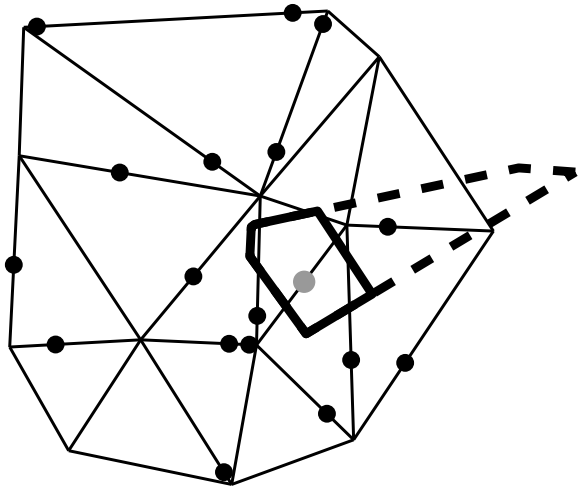

Cell cut by bisectors

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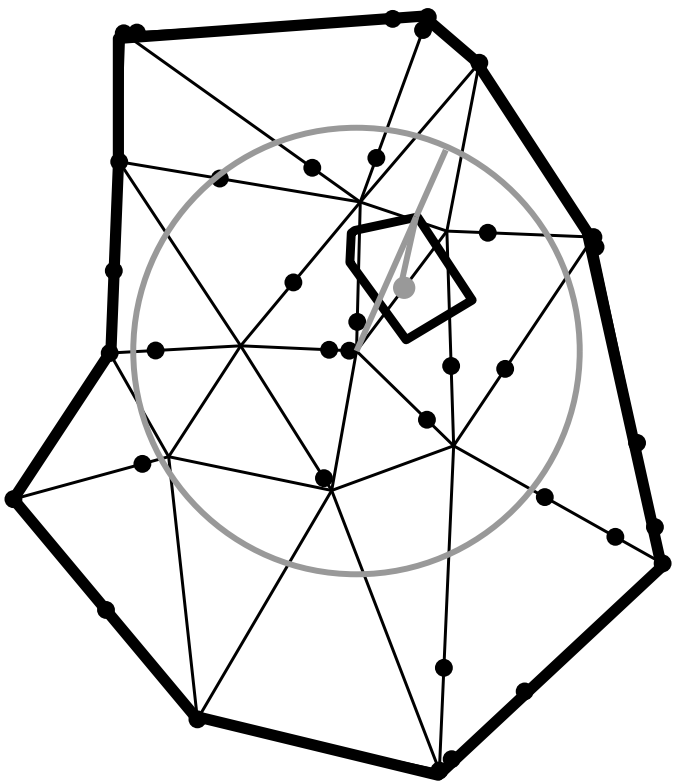
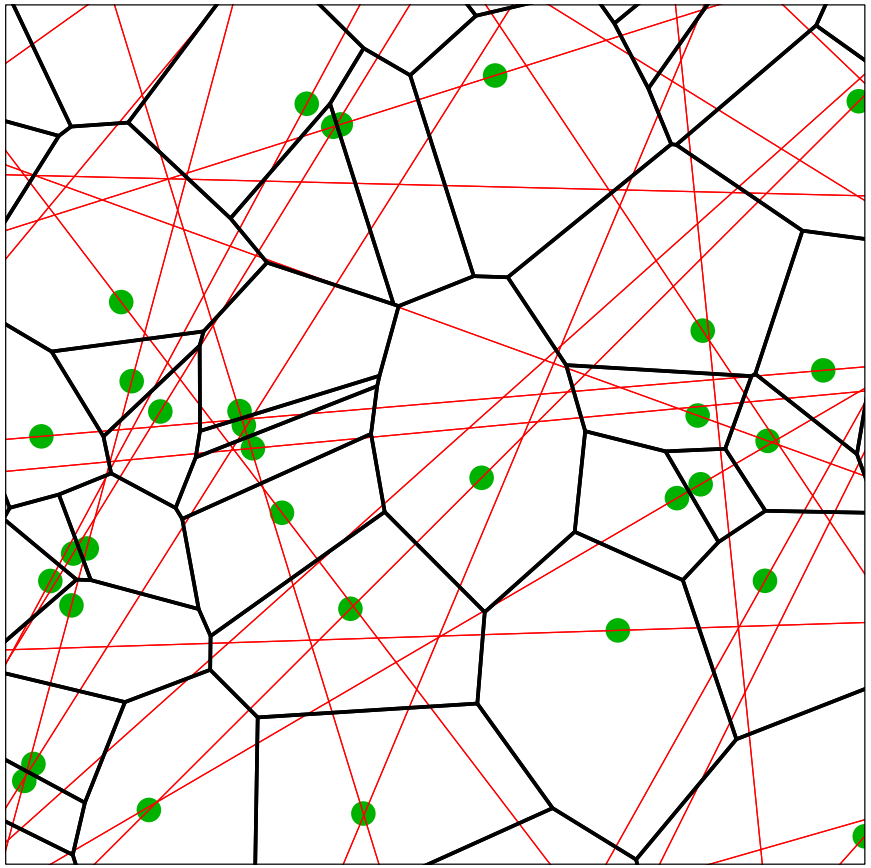


Stop: Weight cell characteristic by $\nu_1(E^*)/\mathbb{E}\nu_1(E^*) = \nu_1(E^*)/(64/(3\pi\sqrt{\lambda}))$

Cox processes on Poisson line tessellations



Cox process on PLT and its Voronoi tessellation

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Theorem

Let

- $T^{(1)}$ *the edge set of a stationary PLT of intensity* γ,
- $\ell(\Phi)$ *the line with* $o \in \ell(\Phi)$ *and direction* $\Phi \sim U[0, \pi)$ *independent of* $T^{(1)}$,
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Thank you for your attention!