Mathematical tools for analysis, modeling and simulation of spatial networks on various length scales

Part I

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Contents

Introduction

Point processes and Palm calculus

Random tessellations

Local simulation of typical Voronoi cells

Cox processes on random tessellations

Multiscale network modeling (Outlook to part II)
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- **Aim**: Stochastic modeling of networks for
  - description of networks by only a few parameters
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  ► simulation of present and future network design scenarios
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- Denote by $\mathbb{N}$ the set of **locally finite counting measures**
  \[ \varphi : \mathcal{B}(\mathbb{R}^2) \to \{0, 1, \ldots \} \cup \{\infty\} \]
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**Examples:**

- Let $\{x_1, \ldots, x_n\} \subset \mathbb{R}^2$, then

$$\varphi(B) = \sum_{i=1}^{n} \delta_{x_i}(B) = \# \{ x_i \in B \} \quad \text{for} \quad B \in \mathcal{B}(\mathbb{R}^2)$$
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- Let $\{x_{i,j} = (i, j) : i \in \mathbb{Z}, j \in \mathbb{Z}\}$, then

$$\varphi(B) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \delta_{(i,j)}(B) = \# \{ x \in \mathbb{Z}^2 \cap B \} \quad \text{for} \quad B \in \mathcal{B}(\mathbb{R}^2)$$
Point processes

Definition

Let

1. $(\Omega, \mathcal{A}, \mathbb{P})$ some probability space,
2. $X_1, X_2, \ldots : \Omega \rightarrow \mathbb{R}^2$ a sequence of random vectors such that

$$\#\{X_n \in B\} < \infty \quad \text{for each bounded} \quad B \in \mathcal{B}(\mathbb{R}^2).$$
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Then the mapping $X$ from $(\Omega, \mathcal{A}, \mathbb{P})$ into $(\mathbb{N}, \mathcal{N})$ defined by $X = \sum_{n=1}^{\infty} \delta_{X_n}$ is called a (random) **point process**.
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- Henceforth: Identify $X = \sum_{n=1}^{\infty} \delta X_n$ and $\{X_n\}$, where we write $X = \{X_n\}$.
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- Henceforth: Identify $X = \sum_{n=1}^{\infty} \delta_{X_n}$ and $\{X_n\}$, where we write $X = \{X_n\}$
- In other words: The point process $X = \{X_n\}$ is identified with the random counting measure $X = \sum_{n=1}^{\infty} \delta_{X_n}$. 
Intensity measure, stationarity and Palm distribution

Definition

Let $X$ be a point process, then

- the **intensity measure** $\mu : \mathcal{B}(\mathbb{R}^2) \to [0, \infty]$ of $X$ is defined as

$$\mu(B) = \mathbb{E}X(B), \quad B \in \mathcal{B}(\mathbb{R}^2),$$
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- $X$ is called **stationary** if $\{X_n - x\} \overset{D}{=} \{X_n\}$ for each $x \in \mathbb{R}^2$. 

Note that a point process $X$ with distribution $P_{oX}$ is called a Palm version of $X$. 

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- If $X$ is stationary, then $\mu(B) = \lambda \nu_2(B), \quad B \in \mathcal{B}(\mathbb{R}^2)$, for some $\lambda > 0$, which is called the **intensity** of $X$. 

Note that a point process $X$ with distribution $P_0^X$ is called a Palm version of $X$. 

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- The **Palm distribution** $P^o_X : \mathcal{N} \mapsto [0, 1]$ of a stationary point process $X$ with intensity $\lambda$ is defined as
  $$P^o_X(A) = \frac{1}{\lambda} \mathbb{E}\#\{n : X_n \in [0, 1]^2, X(\cdot + X_n) \in A\}, \quad A \in \mathcal{N}$$

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- Note that a point process $X^o$ with distribution $P^o_X$ is called a *Palm version* of $X$. 

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**Additional Note:**

The Palm distribution $P^o_X$ captures the expected number of points in an event set $A$, conditioned on the presence of a typical point at the origin. It is a fundamental concept in the study of stationary point processes, particularly in the context of spatial networks and stochastic geometry.
Examples: Stationary Poisson processes

For any fixed $\lambda > 0$, let $X$ be a point process such that

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Then \( X \) is called a stationary Poisson process with intensity \( \lambda \).
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Realization of a stationary Poisson process
General Poisson processes

For any (locally finite) measure $\mu : \mathcal{B}(\mathbb{R}^2) \rightarrow [0, \infty]$, let $X$ be a point process such that

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Realization of a general (non-stationary) Poisson process
Poisson-related point processes

- Poisson cluster processes
- Poisson hardcore processes

Realizations of a Poisson cluster process (left) and a Poisson hardcore process (right)
Example: Matern-cluster processes

- Constructed from Poisson processes (of cluster centers)
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  - Cluster centers form a stationary Poisson process (with some intensity $\lambda_0$)
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- Constructed from Poisson processes (of cluster centers)
  - **Cluster centers** form a stationary Poisson process (with some intensity $\lambda_0$)
  - **Cluster members** form (independent) stationary Poisson processes with some intensity $\lambda_1$, within discs of some radius $R$ around the cluster centers
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- $\Rightarrow$ Spatial interaction between points (mutual attraction)
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  - Realizations are **clustered point patterns** (with higher spatial variability than in the Poisson case)
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  - Cluster centers form a stationary Poisson process (with some intensity $\lambda_0$)
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- Spatial interaction between points (mutual attraction)
  - Realizations are clustered point patterns (with higher spatial variability that in the Poisson case)

- Three-parametric model with parameters $\lambda_0$, $\lambda_1$ and $R$
Example: Matern-hardcore processes

- Constructed from Poisson processes (by random deletion of points)
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  - Realizations are regular point patterns (with smaller spatial variability that in the Poisson case)
- Two-parametric model with parameters $\lambda$ and $R$
Random measures and Cox processes

Random measures

- Denote by $\mathbf{M}$ the set of all locally finite measures $\eta : \mathcal{B}(\mathbb{R}^2) \to [0, \infty]$
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Cox point processes

A point process $X$ is called a Cox process with random intensity measure $\Lambda$ if

$$P(X(B_1) = k_1, \ldots, X(B_n) = k_n) = \mathbb{E}\left(n \prod_{i=1}^{n} \Lambda(B_i) k_i^k e^{-\Lambda(B_i)}\right),$$

for all pairwise disjoint, bounded $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^2)$ and $k_1, \ldots, k_n \geq 0$.

Conditioning on $\Lambda = \eta$, a Cox process $X$ is a Poisson process with intensity measure $\eta$. 
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- A mapping $\Lambda$ from $(\Omega, \mathcal{A}, \mathbb{P})$ into $(\mathbb{M}, \mathcal{M})$ is called a random measure.
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- Conditioning on $\Lambda = \eta$, a Cox process $X$ is a Poisson process with intensity measure $\eta$. 

Marked point processes

- Let $\mathbb{M}$ be a Polish space with Borel $\sigma$-algebra $\mathcal{B}(\mathbb{M})$
- **Examples:**
  - $\mathbb{M} = \mathbb{R}$ and $\mathcal{B}(\mathbb{M}) = \mathcal{B}(\mathbb{R})$, 


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    - $\mathcal{F}$ family of closed sets in $\mathbb{R}^2$, and
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    - $\mathcal{F}$ family of closed sets in $\mathbb{R}^2$, and
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- Let $\mathcal{N}_\mathbb{M}$ be the set of all counting measures $\psi : \mathcal{B} \otimes \mathcal{B}(\mathbb{M}) \to \{0, 1, \ldots\} \cup \{\infty\}$ which are locally finite in the first component, i.e., $\psi(B \times \mathbb{M}) < \infty$ for bounded $B \in \mathcal{B}(\mathbb{R}^2)$,
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- and $\mathcal{N}_\mathbb{M}$ the smallest $\sigma$-algebra on $\mathcal{N}_\mathbb{M}$ such that $\psi \to \psi(B \times G)$ is measurable for each bounded $B \in \mathcal{B}(\mathbb{R}^2)$ and $G \in \mathcal{B}(\mathbb{M})$. 

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- $(\Omega, \mathcal{A}, \mathbb{P})$ some probability space,
- $X_1, X_2, \cdots : \Omega \mapsto \mathbb{R}^2$ and $M_1, M_2, \cdots : \Omega \mapsto \mathbb{M}$ two sequences of $\mathbb{R}^2$- and $\mathbb{M}$-valued random variables, respectively, such that

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  \# \{ X_n \in B \} < \infty \quad \text{for each bounded} \quad B \in \mathcal{B}(\mathbb{R}^2).
  \]
- The measurable mapping $X_M : \Omega \mapsto N_M$ defined by $X_M = \sum_{n=1}^{\infty} \delta(X_n, M_n)$ is called a marked point process and $M_n$ is called the mark (or label) of $X_n$. 
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- $X_1, X_2, \ldots : \Omega \mapsto \mathbb{R}^2$ and $M_1, M_2, \ldots : \Omega \mapsto \mathcal{M}$ two sequences of $\mathbb{R}^2$- and $\mathcal{M}$-valued random variables, respectively, such that

$$\# \{ X_n \in B \} < \infty \quad \text{for each bounded} \quad B \in \mathcal{B}(\mathbb{R}^2).$$

- The measurable mapping $X_M : \Omega \mapsto \mathcal{N}_\mathcal{M}$ defined by $X_M = \sum_{n=1}^{\infty} \delta_{(X_n, M_n)}$ is called a marked point process and $M_n$ is called the mark (or label) of $X_n$.
- The intensity measure $\mu : \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(\mathcal{M}) \to [0, \infty]$ of $X_M$ is defined as

$$\mu(B \times C) = \mathbb{E}X_M(B \times C), \quad B \in \mathcal{B}(\mathbb{R}^2), C \in \mathcal{B}(\mathcal{M}).$$
Example: Poisson-Voronoi tessellation

Let \( X = \{X_n\} \) be a stationary Poisson process and consider the Voronoi cell \( \Xi_n \) of \( X_n \):

\[
\Xi_n = \{ x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k| \ \forall k \neq n \}
\]

Then, \( \{(X_n, M_n)\} \), where \( M_n = \Xi_n - X_n \), is a (stationary) marked point process with mark space \( \mathcal{P}^o \).

Realization of a Poisson process \( \{X_n\} \)
Example: Poisson-Voronoi tessellation

- Let $X = \{X_n\}$ be a stationary Poisson process and consider the Voronoi cell $\Xi_n$ of $X_n$:
  \[
  \Xi_n = \{ x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k| \forall k \neq n \}
  \]
- Then, $\{(X_n, M_n)\}$, where $M_n = \Xi_n - X_n$, is a (stationary) marked point process with mark space $\mathcal{P}^o$
Example: Poisson-Voronoi tessellation

Further (stationary) marked point processes associated with \( \{(X_n, \Xi_n - X_n)\} \):

- \( \{(X_n, \nu_2(\Xi_n))\} \) with mark space \([0, \infty)\)
- \( \{(X_n, \nu_1(\partial \Xi_n))\} \) with mark space \([0, \infty)\)

Realization of a Poisson-Voronoi tessellation \( \{(X_n, \Xi_n - X_n)\} \)
Palm mark distribution and Palm distribution

Definition

- $X_M$ is called stationary if $\{(X_n - x, M_n)\} \overset{D}{=} \{(X_n, M_n)\}$ for each $x \in \mathbb{R}^2$. 

Note that $P^*_X(M)({C}) = P^*_{X,M}({N_M \times C})$ for any $C \in B(M)$. 
Palm mark distribution and Palm distribution

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- $X_M$ is called stationary if $\{ (X_n - x, M_n) \} \overset{D}{=} \{ (X_n, M_n) \}$ for each $x \in \mathbb{R}^2$.
- If $X_M$ is stationary, then

\[
\mu(B \times C) = \lambda \nu_2(B) P_{X_M}^*(C), \quad B \in \mathcal{B}(\mathbb{R}^2),
\]
Palm mark distribution and Palm distribution

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- $X_M$ is called **stationary** if $\{(X_n - x, M_n)\} \overset{D}{=} \{(X_n, M_n)\}$ for each $x \in \mathbb{R}^2$.
- If $X_M$ is stationary, then

\[ \mu(B \times C) = \lambda \nu_2(B) P^*_X(C), \quad B \in \mathcal{B}(\mathbb{R}^2), \]

for some $\lambda > 0$, which is called the **intensity** of $X_M$, for any $C \in \mathcal{B}(M)$. 
Palm mark distribution and Palm distribution

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- $X_M$ is called stationary if $\{(X_n - x, M_n)\} \overset{D}{=} \{(X_n, M_n)\}$ for each $x \in \mathbb{R}^2$.
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for some $\lambda > 0$, which is called the intensity of $X_M$.
- and some probability measure $P_{X_M}^*$ on $\mathcal{B}(\mathcal{M})$, which is called the Palm mark distribution of $X_M$. 
Palm mark distribution and Palm distribution

Definition

- $X_M$ is called stationary if $\{(X_n - x, M_n)\} \overset{D}{=} \{(X_n, M_n)\}$ for each $x \in \mathbb{R}^2$.
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\mu(B \times C) = \lambda\nu_2(B) P_{X_M}^*(C), \quad B \in \mathcal{B}(\mathbb{R}^2),
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- for some $\lambda > 0$, which is called the intensity of $X_M$,
- and some probability measure $P_{X_M}^*$ on $\mathcal{B}(\mathbb{M})$, which is called the Palm mark distribution of $X_M$.

- For any stationary $X_M$ with intensity $\lambda \in (0, \infty)$, the Palm distribution $P_{X_M}^o$ of $X_M$ on $\mathcal{N}_M \otimes \mathcal{B}(\mathbb{M})$ is defined as

$$
P_{X_M}^o(A \times C) = \frac{\mathbb{E}\#\{k : X_k \in [0, 1]^2, M_k \in C, \{(X_n - X_k, M_n)\} \in A\}}{\lambda}
$$

for $A \in \mathcal{N}_M$, $C \in \mathcal{B}(\mathbb{M})$. 

Palm mark distribution and Palm distribution

Definition

▶ **$X_M$ is called stationary if** $(X_n - x, M_n) \overset{D}{=} (X_n, M_n)$ **for each** $x \in \mathbb{R}^2$.

▶ **If** $X_M$ **is stationary, then**

\[
\mu(B \times C) = \lambda \nu_2(B) P^*_X(C), \quad B \in \mathcal{B}(\mathbb{R}^2),
\]

▶ **for some** $\lambda > 0$, **which is called the intensity of** $X_M$,

▶ **and some probability measure** $P^*_X$ **on** $\mathcal{B}(M)$, **which is called the Palm mark distribution of** $X_M$.

▶ **For any stationary** $X_M$ **with intensity** $\lambda \in (0, \infty)$, **the Palm distribution** $P^o_X$ **of** $X_M$ **on** $\mathcal{N}_M \otimes \mathcal{B}(M)$ **is defined as**

\[
P^o_X(A \times C) = \frac{\mathbb{E} \# \{ k : X_k \in [0, 1]^2, M_k \in C, \{(X_n - X_k, M_n)\} \in A \}}{\lambda}
\]

**for** $A \in \mathcal{N}_M, C \in \mathcal{B}(M)$.

▶ **Note that** $P^*_X(C) = P^o_X(\mathcal{N}_M \times C)$ **for any** $C \in \mathcal{B}(M)$. 
Typical mark

**Definition**

Let $X_M$ be a stationary marked point process with Palm mark distribution $\mathbb{P}_{X_M}^*$. 

A random variable $M^* : \Omega \rightarrow \mathbb{M}$ distributed according to $\mathbb{P}_{X_M}^*$ is called the **typical mark** of $X_M$. 

\[ \text{E} h (M^*) = \lim_{r \to \infty} \frac{\# \{ n : X_n \in [−r, r]^2 \} \sum_{i : X_i \in [−r, r]^2} h (M_i) \}} {r^2} \text{ almost surely for each measurable } h : M \to [0, \infty). \]
Typical mark

Definition

Let $X_M$ be a stationary marked point process with Palm mark distribution $\mathbb{P}_{X_M}^\star$.

- A random variable $M^\star : \Omega \rightarrow \mathbb{M}$ distributed according to $\mathbb{P}_{X_M}^\star$ is called the **typical mark** of $X_M$.

- If $X_M$ is **ergodic**, then $M^\star$ can be regarded as the mark at a point chosen purely at random out of $\{X_n\}$, i.e.,

$$\mathbb{E} h(M^\star) = \lim_{r \to \infty} \frac{1}{\# \{ n : X_n \in [-r, r]^2 \}} \sum_{i : X_i \in [-r, r]^2} h(M_i)$$

almost surely for each measurable $h : \mathbb{M} \mapsto [0, \infty)$. 
Example: Independent marking

- Let \( X = \{X_N\} \) be a point process and
- \( M_1, M_2, \cdots : \Omega \to \mathbb{R} \) i.i.d. random variables with some distribution \( P \), which are independent of \( \{X_n\} \).
Example: Independent marking

- Let $X = \{X_N\}$ be a point process and
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- Then, $X_M = \{(X_n, M_n)\}$ is called an independently marked point process.
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Then, $X_M = \{(X_n, M_n)\}$ is called an independently marked point process.

Palm version of independently marked point processes

Let $X = \{X_n\}$ be stationary and $X^o = \{X_n^o\}$ a Palm version of $X$ (with distribution $P_X^o$).
Example: Independent marking

- Let \( X = \{X_N \} \) be a point process and
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- Then, \( X_M = \{(X_n, M_n)\} \) is called an independently marked point process.

Palm version of independently marked point processes

- Let \( X = \{X_n\} \) be stationary and \( X^o = \{X^o_n\} \) a Palm version of \( X \) (with distribution \( P^o_X \)).
- If \( X^o \) is independent of \( \{M_n\} \), then
  - the distribution of the marked point process \( X^o_M = \{(X^o_n, M_n)\} \) is given by \( P^o_{X_M} \), i.e., \( X^o_M \) is a Palm version of \( X_M = \{(X_n, M_n)\} \),
Example: Independent marking

- Let \( X = \{X_n\} \) be a point process and
- \( M_1, M_2, \ldots : \Omega \to \mathbb{R} \) i.i.d. random variables with some distribution \( P \), which are independent of \( \{X_n\} \).
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Palm version of independently marked point processes

- Let \( X = \{X_n\} \) be stationary and \( X^o = \{X^o_n\} \) a Palm version of \( X \) (with distribution \( P^o_X \)).
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  - the distribution of the marked point process \( X^o_M = \{(X^o_n, M_n)\} \) is given by \( P^o_{X^o_M} \), i.e., \( X^o_M \) is a Palm version of \( X_M = \{(X_n, M_n)\} \),
  - and the typical mark \( M^* \) of \( X_M = \{(X_n, M_n)\} \) has distribution \( P^*_{X^o_M} = P \).
Example: Poisson-Voronoi tessellation

- Let $X = \{X_n\}$ be a stationary Poisson process.
Example: Poisson-Voronoi tessellation

Let $X = \{X_n\}$ be a stationary Poisson process.

Consider the Voronoi cells $\Xi_n = \{x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k| \forall k \neq n\}$,
Example: Poisson-Voronoi tessellation

- Let $X = \{X_n\}$ be a stationary Poisson process.
- Consider the Voronoi cells $\Xi_n = \{x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k| \, \forall k \neq n\}$,
- and the stationary marked point process $X_M = \{(X_n, \Xi_n - X_n)\}$

Realization of the Poisson-Voronoi tessellation $\{(X_n, \Xi_n - X_n)\}$
Example: Poisson-Voronoi tessellation

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Realization of the Poisson-Voronoi tessellation $\{(X_n, \Xi_n - X_n)\}$

Palm version of the Poisson-Voronoi tessellation $\{(X_n, \Xi_n - X_n)\}$

- Add the origin $X_0 = o$ to the stationary Poisson process $X = \{X_n\}$. 
Example: Poisson-Voronoi tessellation

Then, by Slivnyak’s theorem, the point process $X^o = \{X_n^o\}$, where $\{X_n^o\} = \{X_0, X_1, X_2, \ldots\}$, is a Palm version of $X = \{X_1, X_2, \ldots\}$. 

Realization of the Palm version $\{(X_n^o, \Xi_n^o - X_n^o)\}$

The typical mark $\Xi^*$ of $X_M$ is given by $\Xi^* = \Xi_{o0}$. 

Example: Poisson-Voronoi tessellation

Then, by Slivnyak’s theorem, the point process $X^o = \{ X^o_n \}$, where $\{ X^o_n \} = \{ X_0, X_1, X_2, \ldots \}$, is a Palm version of $X = \{ X_1, X_2, \ldots \}$.

Consider the Voronoi cells $\Xi^o_n$ induced by $X^o$, where

$$\Xi^o_n = \{ x \in \mathbb{R}^2 : |x - X^o_n| \leq |x - X^o_k| \ \forall k \in \{0, 1, 2, \ldots \}, \ k \neq n \}.$$
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Then, the marked point process $X^o_M = \{(X^o_n, \Xi^o_n - X^o_n)\}$ is a Palm version of $X_M = \{(X_n, \Xi_n - X_n)\}$.

Realization of the Palm version $\{(X^o_n, \Xi^o_n - X^o_n)\}$
Example: Poisson-Voronoi tessellation

- Then, by Slivnyak’s theorem, the point process $X^o = \{X^o_n\}$, where $\{X^o_n\} = \{X_0, X_1, X_2, \ldots\}$, is a Palm version of $X = \{X_1, X_2, \ldots\}$.

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**Realization of the Palm version $\{(X^o_n, \Xi^o_n - X^o_n)\}$**

- The typical mark $\Xi^*$ of $X_M$ is given by $\Xi^* = \Xi^o_0$
Contents

Introduction

Point processes and Palm calculus

Random tessellations

Local simulation of typical Voronoi cells

Cox processes on random tessellations

Multiscale network modeling (Outlook to part II)
Random tessellations

General idea

- Tessellation
  - countable (locally finite) subdivision of $\mathbb{R}^2$
Random tessellations

General idea

- Tessellation
  - countable (locally finite) subdivision of $\mathbb{R}^2$
  - into non-overlapping closed sets (with non-empty interiors), called cells
Random tessellations

General idea

▷ Tessellation
  ▷ countable (locally finite) subdivision of $\mathbb{R}^2$
  ▷ into non-overlapping closed sets (with non-empty interiors), called **cells**

▷ Random tessellation
  ▷ Random marked point process $T = \{X_n, M_n\}$ with mark space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$, ...
Random tessellations

General idea

- **Tessellation**
  - countable (locally finite) subdivision of $\mathbb{R}^2$
  - into non-overlapping closed sets (with non-empty interiors), called *cells*

- **Random tessellation**
  - Random marked point process $T = \{X_n, M_n\}$ with mark space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$,
  - where $\mathcal{F}$ = the family of all closed sets in $\mathbb{R}^2$, and
Random tessellations

General idea

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Random tessellations

General idea

- **Tessellation**
  - countable (locally finite) **subdivision** of $\mathbb{R}^2$
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Examples

- **Tessellations with convex cells**
  - Voronoi tessellations
  - Laguerre tessellations (generalization of Voronoi tessellations)
  - Delaunay tessellations
  - line tessellations
Random tessellations

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Examples

- **Tessellations with convex cells**
  - Voronoi tessellations
  - Laguerre tessellations (generalization of Voronoi tessellations)
  - Delaunay tessellations
  - line tessellations

- **Tessellations with general (not necessarily convex) cells**
  - aggregate tessellations
  - generalized Laguerre tessellations
  - $\beta$-skeletons (thinnings of Delaunay tessellations)
Poisson-Voronoi tessellation

- Cells are generated by a point process \( \{X_n\} \)
Poisson-Voronoi tessellation

- Cells are generated by a point process \(\{X_n\}\)
- Cell \(\Xi_n\) of point \(X_n\) is given by
  \[
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- If \( \{X_n\} \) stationary Poisson point process
  \( \Rightarrow \) Poisson-Voronoi tessellation (PVT)

Realization of a PVT \( \{(X_n, \Xi_n - X_n)\} \)
Poisson-Laguerre tessellation

Let $X_R = \{(X_n, R_n)\}$ a marked point process with non-negative marks $R_n$. 
Poisson-Laguerre tessellation

Let $X_R = \{(X_n, R_n)\}$ a marked point process with non-negative marks $R_n$.

The Laguerre cell $\Xi_n$ of $X_n$ is given by

$$\Xi_n = \{ x \in \mathbb{R}^2 : |x - X_n|^2 - R_n^2 \leq |x - X_k|^2 - R_k^2, \forall k \neq n \}$$
Poisson-Laguerre tessellation

Let $X_R = \{(X_n, R_n)\}$ a marked point process with non-negative marks $R_n$.

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Then, $T = \{(X_n, \Xi_n - X_n) \text{ such that } \text{int}(\Xi_n) \neq \emptyset\}$ is called a Laguerre tessellation induced by $X_R = \{(X_n, R_n)\}$,
Poisson-Laguerre tessellation

- Let \( X_R = \{(X_n, R_n)\} \) a marked point process with non-negative marks \( R_n \).
- The Laguerre cell \( \Xi_n \) of \( X_n \) is given by
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  \]

- Then, \( T = \{(X_n, \Xi_n - X_n) : \text{int}(\Xi_n) \neq \emptyset \} \) is called
  - a Laguerre tessellation induced by \( X_R = \{(X_n, R_n)\} \),
  - which specifies to a Voronoi tessellation if \( R_1 = R_2 = \ldots \)
Poisson-Laguerre tessellation

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- which specifies to a Voronoi tessellation if $R_1 = R_2 = \ldots$

Note that

- the generating point $X_n$ is not necessarily inside the cell $\Xi_n$, and
Poisson-Laguerre tessellation

- Let $X_R = \{(X_n, R_n)\}$ a marked point process with non-negative marks $R_n$.
- The Laguerre cell $\Xi_n$ of $X_n$ is given by
  \[ \Xi_n = \{ x \in \mathbb{R}^2 : |x - X_n|^2 - R_n^2 \leq |x - X_k|^2 - R_k^2, \forall k \neq n \} \]
- Then, $T = \{(X_n, \Xi_n - X_n) \text{ such that } \text{int}(\Xi_n) \neq \emptyset \}$ is called
  - a Laguerre tessellation induced by $X_R = \{(X_n, R_n)\}$,
  - which specifies to a Voronoi tessellation if $R_1 = R_2 = \ldots$
- Note that
  - the generating point $X_n$ is not necessarily inside the cell $\Xi_n$, and
  - a point $X_n$ does not necessarily generate a cell (because $\text{int}(\Xi_n)$ can be empty)
Poisson-Laguerre tessellation

Let $X_R = \{(X_n, R_n)\}$ a marked point process with non-negative marks $R_n$.

The Laguerre cell $\Xi_n$ of $X_n$ is given by

$$\Xi_n = \{x \in \mathbb{R}^2 : |x - X_n|^2 - R_n^2 \leq |x - X_k|^2 - R_k^2, \forall k \neq n\}$$

Then, $T = \{(X_n, \Xi_n - X_n) \text{ such that } \operatorname{int}(\Xi_n) \neq \emptyset\}$ is called

- a Laguerre tessellation induced by $X_R = \{(X_n, R_n)\}$,
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Note that

- the generating point $X_n$ is not necessarily inside the cell $\Xi_n$, and
- a point $X_n$ does not necessarily generate a cell (because $\operatorname{int}(\Xi_n)$ can be empty)

If $X_R = \{(X_n, R_n)\}$ is an independently marked (stationary) Poisson process, then

$$T = \{(X_n, \Xi_n - X_n) \text{ such that } \operatorname{int}(\Xi_n) \neq \emptyset\}$$

is called a Poisson-Laguerre tessellation.
Poisson-Laguerre tessellation

Cutout of Voronoi tessellation (left) and cutout of Laguerre tessellation on the same set of seed points (right)
Poisson-Delaunay tessellation

- Consider a Voronoi tessellation \( T = \{ (X_n, \Xi_n - X_n) \} \) induced by a stationary Poisson process \( \{X_n\} \)
Poisson-Delaunay tessellation

Consider a Voronoi tessellation $T = \{(X_n, \Xi_n - X_n)\}$ induced by a stationary Poisson process $\{X_n\}$.

For each vertex $X'_n$ of $T$ construct the cell $\Xi'_n$ as the triangle formed by the nuclei $X_{i_1}, X_{i_2}, X_{i_3}$ of the three neighboring Voronoi cells.
Poisson-Delaunay tessellation

Consider a Voronoi tessellation $T = \{(X_n, \Xi_n - X_n)\}$ induced by a stationary Poisson process $\{X_n\}$.

For each vertex $X'_n$ of $T$ construct the cell $\Xi'_n$ as the triangle formed by the nuclei $X_{i_1}, X_{i_2}, X_{i_3}$ of the three neighboring Voronoi cells.

Then $T' = \{(X'_n, \Xi'_n - X'_n)\}$ is called a Poisson-Delaunay tessellation.
Poisson line tessellation

Let

- $\{R_n\}$ a stationary Poisson process on the real line $\mathbb{R}$
Poisson line tessellation

Let

- $\{R_n\}$ a stationary Poisson process on the real line $\mathbb{R}$
- $\{\Phi_n\}$ i.i.d. r.v.'s, independent of $\{R_n\}$, with $\Phi_n \sim U[0, \pi)$, and
Poisson line tessellation

Let

- \( \{ R_n \} \) a stationary Poisson process on the real line \( \mathbb{R} \)
- \( \{ \Phi_n \} \) i.i.d. r.v.’s, independent of \( \{ R_n \} \), with \( \Phi_n \sim \mathbb{U}[0, \pi) \), and
- \( \ell(\Phi_n, R_n) = \{(x, y) \in \mathbb{R}^2 : x \sin \Phi_n - y \cos \Phi_n = R_n \} \) the line with direction \( \Phi_n \) and signed distance \( R_n \) to the origin \( o \in \mathbb{R}^2 \).
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Then, \{\ell(\Phi_n, R_n)\} is called a Poisson line process, where
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Then, \( \{\ell(\Phi_n, R_n)\} \) is called a Poisson line process, where \( T^{(1)} = \bigcup_{n \in \mathbb{Z}} \ell(\Phi_n, R_n) \) is the edge set of a Poisson line tessellation (PLT).

Realization of a Poisson line tessellation
Tessellations with general (not necessarily convex) cells

- Aggregate Voronoi tessellations

Construction principle (left) and cutout of an aggregate tessellation (right)
Generalized Laguerre tessellations

Let $X = \{(X_n, [R_n, A_n])\}$ be a marked point process, where

- the $R_n$ are non-negative r.v.'s, and
Generalized Laguerre tessellations

Let \( X_R = \{ (X_n, [R_n, A_n]) \} \) be a marked point process, where
- the \( R_n \) are non-negative r.v.’s, and
- the \( A_n \) are positive definite random \( 2 \times 2 \)-matrices.
Generalized Laguerre tessellations

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The generalized Laguerre cell $\Xi_n$ of $X_n$ is given by

$$
\Xi_n = \{x \in \mathbb{R}^2 : |x - X_n|_{A_n}^2 - R_n^2 \leq |x - X_k|_{A_n}^2 - R_k^2, \forall k \neq n\},
$$

where $|x|_A = \sqrt{x^T A x}$ for all $x \in \mathbb{R}^2$. 

Note that

- the generating point $X_n$ is not necessarily inside the cell $\Xi_n$, and
- a point $X_n$ does not necessarily generate a cell (because $\text{int}(\Xi_n)$ can be empty)
- the cells $\Xi_n$ are not necessarily convex.
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where $|x|_A = \sqrt{x^\top A x}$ for all $x \in \mathbb{R}^2$.

Then, $T = \{(X_n, \Xi_n - X_n) \text{ such that } \text{int}(\Xi_n) \neq \emptyset \}$ is called
- a generalized Laguerre tessellation induced by $X_R = \{(X_n, [R_n, A_n])\}$,
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  - which specifies to a Laguerre tessellation if $A_1 = A_2 = \ldots = I$ and to a
    Voronoi tessellation if $A_1 = A_2 = \ldots = I$ and $R_1 = R_2 = \ldots$. 
Generalized Laguerre tessellations

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Generalized Laguerre tessellations

Seed points $X_n$, radii $R_n$, ellipse-representation of matrices $A_n$ (left), and cutout of generalized Laguerre tessellation (right)
$\beta$-skeletons

- Let $\beta \in [1, 2]$ any fixed number.
\textbf{\(\beta\)-skeletons}

\begin{itemize}
  \item Let \(\beta \in [1, 2]\) any fixed number.
  \item For \(x, y \in \mathbb{R}^2\) consider the \textit{weighted means}
\end{itemize}

\[
m_{xy}^{(1)} = \frac{\beta}{2} x + (1 - \frac{\beta}{2}) y, \quad m_{xy}^{(2)} = (1 - \frac{\beta}{2}) x + \frac{\beta}{2} y,
\]
**β-skeletons**

- Let $\beta \in [1, 2]$ any fixed number.
- For $x, y \in \mathbb{R}^2$ consider the **weighted means**
  
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  and the **intersection of two balls**
  
  $$A_\beta(x, y) = B(m_{xy}^{(1)}, |m_{xy}^{(1)} - y|) \cap B(m_{xy}^{(2)}, |m_{xy}^{(2)} - x|).$$
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Illustration of the intersection $A_\beta(x, y)$ of the two balls:
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Illustration of the intersection \(A_\beta(x, y)\) of the two balls:
for \(\beta = 1\) (dotted), \(\beta = 1.5\) (dashed) and \(\beta = 2\) (solid)
**β-skeletons**

- Let $\beta \in [1, 2]$ any fixed number and $X = \{X_n\}$ a point process in $\mathbb{R}^2$. Then, the edge set $G(\beta, X) = \bigcup_{x, y \in X} X \cap A_\beta(x, y)$ is called a $\beta$-skeleton induced by $X = \{X_n\}$. Examples of $\beta$-skeletons for $\beta = 1$, $\beta = 1.5$ and $\beta = 2$ (left to right). Note that the edge set $G(\beta, I)$ is monotonously decreasing in $\beta$, and for $\beta = 1$, $\beta$-skeletons specify to the edge sets of Delaunay tessellations.
**β-skeletons**

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$$G(\beta, X) = \bigcup_{x, y \in X: X \cap A_\beta(x, y) = \emptyset} [x, y]$$

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Contents

Introduction

Point processes and Palm calculus

Random tessellations

Local simulation of typical Voronoi cells

Cox processes on random tessellations

Multiscale network modeling (Outlook to part II)
Local simulation of the typical Poisson-Voronoi cell

General idea

- Consider a stationary Poisson process $X$ with some intensity $\lambda > 0$. 

Local simulation of the typical Poisson-Voronoi cell

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  \[ \{X_1, X_2, \ldots, X_n\} = X \cup \{o\}. \]
- Use a suitable stopping rule to reduce runtime.
Radial simulation of Poisson processes

Theorem

Let
- \( \lambda > 0 \) be an arbitrary, but fixed number,
- \( Y_1, Y_2, \ldots \) i.i.d. \( \text{Exp}(1) \)-distributed,
- \( R_n = \sqrt{\sum_{k=1}^{n} \frac{Y_k}{\pi \lambda}} \) for \( n = 1, 2, \ldots \),
- \( U_1, U_2, \ldots \) i.i.d. \( \text{U}[0, 2\pi) \)-distributed and
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Then \( \{X_n\} \) is a stationary Poisson process in \( \mathbb{R}^2 \) with intensity \( \lambda \).
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Then \( \{X_n\} \) is a **stationary Poisson process** in \( \mathbb{R}^2 \) with intensity \( \lambda \).

Proof  Idea: Show that \( X(B) \sim \text{Poi}(\lambda \nu_2(B)) \) and \( X(B_1), \ldots, X(B_n) \) are independent for \( B_1, \ldots, B_n \in \mathcal{B} \) with \( B_i \cap B_j = \emptyset \) for \( i \neq j \), e.g.,
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- $X_n = (R_n \cos U_n, R_n \sin U_n)$ for $n = 1, 2, \ldots$.

Then $\{X_n\}$ is a stationary Poisson process in $\mathbb{R}^2$ with intensity $\lambda$.

**Proof** Idea: Show that $X(B) \sim Pois(\lambda \nu_2(B))$ and $X(B_1), \ldots, X(B_n)$ are independent for $B_1, \ldots, B_n \in \mathcal{B}$ with $B_i \cap B_j = \emptyset$ for $i \neq j$, e.g.,

$X(B(o, r)) = \#\{n : R_n \leq r\} = \#\{n : \sum_{k=1}^{n} Y_k \leq \lambda \pi r^2\} \sim Pois(\lambda \pi r^2)$. 
Radial simulation of Poisson processes

**Algorithm:**

- Simulate $Y_n \sim \text{Exp}(1), U_n \sim U[0, 2\pi)$ independent of $Y_1, \ldots, Y_{n-1}, U_1, \ldots, U_{n-1}$
- Construct $X_n = (R_n \cos U_n, R_n \sin U_n)$ with $R_n = \sqrt{\sum_{k=1}^{n} Y_k / (\pi \lambda)}$
Radial simulation of Poisson processes

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  - Construct $X_n = (R_n \cos U_n, R_n \sin U_n)$ with $R_n = \sqrt{\sum_{k=1}^{n} Y_k / (\pi \lambda)}$

- Stop if $R_n > a/\sqrt{2}$, where $a$ is the side length of the sampling window
**Slivnyak’s theorem**

**Theorem**

*Let $X$ be a stationary Poisson process with some intensity $\lambda > 0$. Then*

$$
P(X^0 \in A) = P(X \cup \{o\} \in A),
$$

*where $X^0$ is the Palm version of $X$, i.e., $X^0$ is distributed according to the Palm distribution $P_X^0$ of $X$.***
**Slovinak’s theorem**

**Theorem**

Let $X$ be a stationary Poisson process with some intensity $\lambda > 0$. Then

$$P(X^0 \in A) = P(X \cup \{o\} \in A),$$

where $X^0$ is the **Palm version** of $X$, i.e., $X^0$ is distributed according to the Palm distribution $P^0_X$ of $X$.

**Proof** Consider void probabilities $P(X^0(C) = 0), C \subset \mathbb{R}^2$ compact. Then $P(X^0(\{o\}) = 1) = P(X(\{o\}) = 0) = 1$ by definition. Furthermore, if $o \notin C$, then

$$P(X^0 \cap C = 0) = \lim_{\varepsilon \searrow 0} P(X(C) = 0 \mid X(B(o, \varepsilon)) = 1)$$
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\[
P(X^0C) = 0) = \lim_{\varepsilon \searrow 0} P(X(C) = 0 \mid X(B(o, \varepsilon)) = 1)
\]

\[
= \lim_{\varepsilon \searrow 0} \frac{P(X(C) = 0)P(X(B(o, \varepsilon) \setminus C) = 1)}{P(X(B(o, \varepsilon)) = 1)
\]

\[
= P(X(C) = 0) .
\]
The typical Voronoi cell

Let

\[ \{X_n\} \text{ be a stationary Poisson process and } T = \{\Xi_n\} \text{ the induced Poisson-Voronoi tessellation (PVT), i.e.,} \]

\[ \Xi_n = \{x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k| \forall k \neq n\} \]

\[ = \bigcap_{k \in \mathbb{N} : k \neq n} H(X_n, X_k) \]

with half planes \( H(X_n, X_k) = \{x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k|\} \)
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= \bigcap_{k \in \mathbb{N} : k \neq n} H(X_n, X_k)
\]

with half planes \(H(X_n, X_k) = \{x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k|\}\)

\(\Xi^*\) be the typical cell of \(T\), i.e., \(\Xi^*\) is the typical mark of \(\{(X_n, \Xi_n - X_n)\}\)
The typical Voronoi cell

- Let
  - \( \{ X_n \} \) be a stationary Poisson process and \( T = \{ \Xi_n \} \) the induced Poisson-Voronoi tessellation (PVT), i.e.,
    \[
    \Xi_n = \{ x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k| \quad \forall k \neq n \}
    \]
    \[
    = \bigcap_{k \in \mathbb{N} : k \neq n} H(X_n, X_k)
    \]
  with half planes \( H(X_n, X_k) = \{ x \in \mathbb{R}^2 : |x - X_n| \leq |x - X_k| \} \)

- \( \Xi^* \) be the typical cell of \( T \), i.e., \( \Xi^* \) is the typical mark of \( \{(X_n, \Xi_n - X_n)\} \)

- Slivnyak’s theorem yields

\[
\Xi^* = \bigcap_{n=1}^{\infty} H(o, X_n)
\]
Local simulation of the typical cell of PVT

**Algorithm:**
- Place point at $o$
- Simulate points $X_1, X_2, X_3$ of Poisson process $X$ radially
Local simulation of the typical cell of PVT

- **Algorithm:**
  - Intersect halfplanes $H(o, X_1)$, $H(o, X_2)$ and $H(o, X_3)$
Local simulation of the typical cell of PVT

Algorithm:

- Simulate further points of $X$ and intersect halfplanes $\Rightarrow$ initial cell
Local simulation of the typical cell of PVT

Algorithm:
- Simulate further points of $X$ and intersect initial cell with halfplanes
Local simulation of the typical cell of PVT

- **Algorithm:**
  - Simulate further points of $X$ and intersect initial cell with halfplanes
Local simulation of the typical cell of PVT

Algorithm:

- Stop if $|X_n| > 2 \max_{1 \leq i \leq m} |V_m|$, where $V_1, \ldots, V_m$ are the vertices of the current modification of the initial cell.
Contents

Introduction

Point processes and Palm calculus

Random tessellations

Local simulation of typical Voronoi cells

Cox processes on random tessellations

Multiscale network modeling (Outlook to part II)
Cox processes on random tessellations

Let

- $\lambda_\ell > 0$ any fixed number,
- $T$ a random tessellation with edge set $T^{(1)}$. 

Then, $X$ is called a Cox process on $T^{(1)}$ with linear intensity $\lambda_\ell$.

If $T$ is stationary with $\gamma = \nu_1(T^{(1)} \cap [0,1)^2)$, then $X$ is stationary with intensity $\lambda = \lambda_\ell \gamma$.

Let $X$ be a Cox process on $T^{(1)}$.

Then, $X$ is a (conditional) Poisson process with intensity measure $\mu(\cdot) = \lambda_\ell \nu_1(\cdot \cap T^{(1)})$ given $T^{(1)}$ and the points of $X$ are placed as linear Poisson processes of intensity $\lambda_\ell$ on the edges of $T^{(1)}$. 
Cox processes on random tessellations

Let

- $\lambda_\ell > 0$ any fixed number,
- $T$ a random tessellation with edge set $T^{(1)}$.
- $\Lambda$ a random measure with $\Lambda(B) = \lambda_\ell \nu_1(B \cap T^{(1)})$ for $B \in \mathcal{B}(\mathbb{R}^2)$.
Cox processes on random tessellations

Let

- \( \lambda_\ell > 0 \) any fixed number,
- \( T \) a random tessellation with edge set \( T^{(1)} \).
- \( \Lambda \) a random measure with \( \Lambda(B) = \lambda_\ell \nu_1(B \cap T^{(1)}) \) for \( B \in \mathcal{B}(\mathbb{R}^2) \).
- \( X \) the Cox process with random intensity measure \( \Lambda \).
Cox processes on random tessellations

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- \( \lambda \ell > 0 \) any fixed number,
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Then, \( X \) is called a Cox process on \( T^{(1)} \) with linear intensity \( \lambda \ell \).
Cox processes on random tessellations

Let

- $\lambda_\ell > 0$ any fixed number,
- $T$ a random tessellation with edge set $T^{(1)}$.
- $\Lambda$ a random measure with $\Lambda(B) = \lambda_\ell \nu_1(B \cap T^{(1)})$ for $B \in \mathcal{B}(\mathbb{R}^2)$.
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Cox processes on random tessellations

- Let
  - $\lambda_\ell > 0$ any fixed number,
  - $T$ a random tessellation with edge set $T^{(1)}$.
  - $\Lambda$ a random measure with $\Lambda(B) = \lambda_\ell \nu_1(B \cap T^{(1)})$ for $B \in \mathcal{B}(\mathbb{R}^2)$.
  - $X$ the Cox process with random intensity measure $\Lambda$.

  Then, $X$ is called a Cox process on $T^{(1)}$ with linear intensity $\lambda_\ell$.

- If $T$ is stationary with $\gamma = \nu_1(T^{(1)} \cap [0, 1)^2)$, then $X$ is stationary with intensity $\lambda = \lambda_\ell \gamma$.

- Let $X$ be a Cox process on $T^{(1)}$
  - Then, $X$ is a (conditional) Poisson process with intensity measure $\mu(\cdot) = \lambda_\ell \nu_1(\cdot \cap T^{(1)})$ given $T^{(1)}$. 
Cox processes on random tessellations

Let

- $\lambda_\ell > 0$ any fixed number,
- $T$ a random tessellation with edge set $T^{(1)}$.
- $\Lambda$ a random measure with $\Lambda(B) = \lambda_\ell \nu_1(B \cap T^{(1)})$ for $B \in \mathcal{B}(\mathbb{R}^2)$.
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Then, $X$ is called a Cox process on $T^{(1)}$ with linear intensity $\lambda_\ell$.

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- Then, $X$ is a (conditional) Poisson process with intensity measure $\mu(\cdot) = \lambda_\ell \nu_1(\cdot \cap T^{(1)})$ given $T^{(1)}$
- and the points of $X$ are placed as linear Poisson processes of intensity $\lambda_\ell$ on the edges of $T^{(1)}$
Cox processes on random tessellations

Examples

Realizations of Cox processes on the edge sets of various random tessellations
Local simulation of typical Cox-Voronoï cells

General idea
Local simulation of typical Cox-Voronoï cells

General idea

- Consider a stationary Cox process $X$ whose random intensity measure $\Lambda$ is concentrated on the edge set $T^{(1)}$ of a stationary tessellation $T$. 
Local simulation of typical Cox-Voronoi cells

General idea

► Consider a stationary Cox process $X$ whose random intensity measure $\Lambda$ is concentrated on the edge set $T^{(1)}$ of a stationary tessellation $T$.
► Use Slivnyak’s theorem, which stays that
  ► the Palm version $X^0$ of $X$ is a Cox process
Local simulation of typical Cox-Voronoï cells

General idea

- Consider a stationary Cox process $X$ whose random intensity measure $\Lambda$ is concentrated on the edge set $T^{(1)}$ of a stationary tessellation $T$.
- Use **Slivnyak’s theorem**, which stays that
  - the Palm version $X^0$ of $X$ is a Cox process
  - whose random intensity measure $\Lambda^0$ is concentrated on the edge set $\tilde{T}^{(1)}$ of a conditional version $\tilde{T}$ of $T$, given that $o \in T^{(1)}$. 
Local simulation of typical Cox-Voronoï cells

General idea

- Consider a stationary Cox process $X$ whose random intensity measure $\Lambda$ is concentrated on the edge set $T^{(1)}$ of a stationary tessellation $T$.
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  - whose random intensity measure $\Lambda^0$ is concentrated on the edge set $\tilde{T}^{(1)}$ of a conditional version $\tilde{T}$ of $T$, given that $o \in T^{(1)}$.
- Use a suitable representation of $\tilde{T}$. 
Local simulation of typical Cox-Voronoï cells

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- Use a suitable representation of $\tilde{T}$.

Then,

- simulate the underlying tessellation $\tilde{T}$ (under the condition that $o \in T^{(1)}$) and points of the Cox process on $\tilde{T}^{(1)}$ (approximatively) radially,
Local simulation of typical Cox-Voronoï cells

General idea

- Consider a stationary Cox process $X$ whose random intensity measure $\Lambda$ is concentrated on the edge set $T^{(1)}$ of a stationary tessellation $T$.
- Use Slivnyak’s theorem, which stays that
  - the Palm version $X^0$ of $X$ is a Cox process
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Then,

- simulate the underlying tessellation $\tilde{T}$ (under the condition that $o \in T^{(1)}$) and points of the Cox process on $\tilde{T}^{(1)}$ (approximatively) radially,
- add new edges of $\tilde{T}$ and new points of the Cox process on $\tilde{T}^{(1)}$ in an alternating fashion.
Local simulation of typical Cox-Voronoi cells

General idea

- Consider a stationary Cox process $X$ whose random intensity measure $\Lambda$ is concentrated on the edge set $T^{(1)}$ of a stationary tessellation $T$.
- Use Slivnyak’s theorem, which stays that
  - the Palm version $X^0$ of $X$ is a Cox process
  - whose random intensity measure $\Lambda^0$ is concentrated on the edge set $\tilde{T}^{(1)}$ of a conditional version $\tilde{T}$ of $T$, given that $o \in T^{(1)}$.
- Use a suitable representation of $\tilde{T}$.

Then,

- simulate the underlying tessellation $\tilde{T}$ (under the condition that $o \in T^{(1)}$) and points of the Cox process on $\tilde{T}^{(1)}$ (approximatively) radially,
- add new edges of $\tilde{T}$ and new points of the Cox process on $\tilde{T}^{(1)}$ in an alternating fashion
- Find a good stopping rule to reduce runtime.
Slivnyak’s theorem for Cox processes

**Theorem**

Let $X$ be a Cox process with stationary random intensity measure $\Lambda$. Then,

$$P(X_0 \in A) = P(\tilde{X}_0 \cup \{o\} \in A),$$

where $\tilde{X}$ is a Cox process whose driving measure is the Palm version $\Lambda_0$ of $\Lambda$.

Example: Let $\Lambda(\cdot) = \lambda \ell \nu_1(\cdot \cap T(1))$ be concentrated on the edge set $T(1)$ of some stationary tessellation $T$ with (length) intensity $\gamma > 0$. Then,

$$P(\Lambda_0(A)) = \frac{1}{\gamma} \mathbb{E} \int_{T(1) \cap [0,1]} I_A(\Lambda(\cdot) + x) \nu_1(dx),$$

$A \in \mathbb{N}$.

Thus, $\Lambda_0$ is given by $\Lambda_0(\cdot) = \lambda \ell \nu_1(\cdot \cap \tilde{T}(1))$, where $\tilde{T}$ can be regarded as the conditional version of $T$ under the condition that $o \in T(1)$.
Slivnyak’s theorem for Cox processes

Theorem

Let $X$ be a Cox process with stationary random intensity measure $\Lambda$. Then, the distribution of the Palm version $X^0$ of $X$ is given by

$$
P(X^0 \in A) = P(\tilde{X} \cup \{o\} \in A),$$

where $\tilde{X}$ is a Cox process whose driving measure is the Palm version $\Lambda^0$ of $\Lambda$. 

Example: Let $\Lambda(\cdot) = \lambda \ell \nu_1(\cdot \cap T(1))$ be concentrated on the edge set $T(1)$ of some stationary tessellation $T$ with (length) intensity $\gamma > 0$. Then, the distribution $P_{\Lambda^0}$ of $\Lambda^0$ is given by

$$
P_{\Lambda^0}(A) = \frac{1}{\gamma} E \int_{T(1) \cap [0,1)^2} 1_A(\Lambda(\cdot + x)) \nu_1(dx),$$

$A \in \mathbb{N}$. Thus, $\Lambda^0$ is given by $\Lambda^0(\cdot) = \lambda \ell \nu_1(\cdot \cap \tilde{T}(1))$, where $\tilde{T}$ can be regarded as conditional version of $T$ under the condition that $o \in T(1)$. 
Slivnyak’s theorem for Cox processes

**Theorem**

Let $X$ be a **Cox process** with stationary random intensity measure $\Lambda$. Then, the distribution of the **Palm version** $X^0$ of $X$ is given by

$$
\mathbb{P}(X^0 \in A) = \mathbb{P}(\tilde{X} \cup \{o\} \in A),
$$

where $\tilde{X}$ is a Cox process whose driving measure is the Palm version $\Lambda^0$ of $\Lambda$. 

**Slivnyak’s theorem for Cox processes**

**Theorem**

Let $X$ be a **Cox process** with stationary random intensity measure $\Lambda$. Then, the distribution of the **Palm version** $X^0$ of $X$ is given by

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Let $X$ be a Cox process with stationary random intensity measure $\Lambda$. Then, the distribution of the Palm version $X^0$ of $X$ is given by

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$\triangleright$ the distribution $P_{\Lambda^0}$ of $\Lambda^0$ is given by

$$P_{\Lambda^0}(A) = \frac{1}{\gamma} \mathbb{E} \int_{T^{(1)} \cap [0,1)^2} \mathbb{1}_A(\Lambda(\cdot + x)) \nu_1(dx), \quad A \in \mathcal{N}.$$
Slivnyak’s theorem for Cox processes

Theorem

Let $X$ be a Cox process with stationary random intensity measure $\Lambda$. Then, the distribution of the Palm version $X^0$ of $X$ is given by

$$\mathbb{P}(X^0 \in A) = \mathbb{P}(\tilde{X} \cup \{o\} \in A),$$

where $\tilde{X}$ is a Cox process whose driving measure is the Palm version $\Lambda^0$ of $\Lambda$.

Example: Let $\Lambda(\cdot) = \lambda\nu_1(\cdot \cap T^{(1)})$ be concentrated on the edge set $T^{(1)}$ of some stationary tessellation $T$ with (length) intensity $\gamma > 0$. Then,

- the distribution $P_{\Lambda^0}$ of $\Lambda^0$ is given by

$$P_{\Lambda^0}(A) = \frac{1}{\gamma} \mathbb{E} \int_{T^{(1)} \cap [0,1)^2} \mathbb{1}_A(\Lambda(\cdot + x)) \nu_1(dx), \quad A \in \mathcal{N}.$$ 

- Thus, $\Lambda^0$ is given by $\Lambda^0(\cdot) = \lambda\nu_1(\cdot \cap \tilde{T}^{(1)})$, where $\tilde{T}$ can be regarded as conditional version of $T$ under the condition that $o \in T^{(1)}$. 

- ▶
Cox processes on Poisson-Voronoi tessellations

Cox process on PVT and its Voronoi tessellation
Representation of $\tilde{T}$ for Poisson-Voronoi tessellations

Theorem

Let $T$ be a PVT with intensity $\gamma = 2\sqrt{\lambda}$ induced by a stationary Poisson process with intensity $\lambda$. Theorem

Let $T$ be a PVT with intensity $\gamma = 2\sqrt{\lambda}$ induced by a stationary Poisson process with intensity $\lambda$. Proof

See Baumstark & Last (2007)
Representation of $\tilde{T}$ for Poisson-Voronoi tessellations

**Theorem**

Let $T$ be a PVT with intensity $\gamma = 2\sqrt{\lambda}$ induced by a stationary Poisson process with intensity $\lambda$. Let $R^2$, $\tilde{R}^2$, and $\Phi$ be independent random variables, where

- $R^2$ **gamma distributed** with parameters 1.5 (shape) and $1/(\lambda \pi)$ (scale),
- $\tilde{R}^2$ **beta distributed** with parameters 1 and $1/2$,
- $\Phi$ **uniformly distributed** on $[0, 2\pi)$.

Proof: See Baumstark & Last (2007)
Representation of $\tilde{T}$ for Poisson-Voronoi tessellations

Theorem

Let $T$ be a PVT with intensity $\gamma = 2\sqrt{\lambda}$ induced by a stationary Poisson process with intensity $\lambda$. Let $R^2$, $\tilde{R}^2$ and $\Phi$ be independent random variables, where

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- $\Phi$ uniformly distributed on $[0, 2\pi)$.

Then $\tilde{T}$ is the Voronoi tessellation induced by $\{X_n\}_{n=1}^{\infty}$, where
Representation of \( \tilde{T} \) for Poisson-Voronoi tessellations

**Theorem**

Let \( T \) be a PVT with intensity \( \gamma = 2\sqrt{\lambda} \) induced by a stationary Poisson process with intensity \( \lambda \). Let \( R^2, \tilde{R}^2 \) and \( \Phi \) be independent random variables, where

- \( R^2 \) gamma distributed with parameters 1.5 (shape) and \( 1/(\lambda \pi) \) (scale),
- \( \tilde{R}^2 \) beta distributed with parameters 1 and \( 1/2 \),
- \( \Phi \) uniformly distributed on \([0, 2\pi)\).

Then \( \tilde{T} \) is the Voronoi tessellation induced by \( \{X_n\}_{n=1}^{\infty} \), where

- \( X_1 \) and \( X_2 \) are given by the points \( X_1 = (\sqrt{R^2 - \tilde{R}^2 R^2}, \tilde{R} R) \) and \( X_2 = (\sqrt{R^2 - \tilde{R}^2 R^2}, -\tilde{R} R) \), respectively, rotated around \( o \) with angle \( \Phi \),

Proof

See Baumstark & Last (2007)
Representation of $\tilde{T}$ for Poisson-Voronoi tessellations

**Theorem**

Let $T$ be a PVT with intensity $\gamma = 2\sqrt{\lambda}$ induced by a stationary Poisson process with intensity $\lambda$. Let $R^2, \tilde{R}^2$ and $\Phi$ be independent random variables, where

- $R^2$ **gamma distributed** with parameters 1.5 (shape) and $1/(\lambda \pi)$ (scale),
- $\tilde{R}^2$ **beta distributed** with parameters 1 and 1/2,
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Then $\tilde{T}$ is the **Voronoi tessellation** induced by $\{X_n\}_{n=1}^{\infty}$, where

- $X_1$ and $X_2$ are given by the points $X_1 = (\sqrt{R^2 - \tilde{R}^2 R^2}, \tilde{R} R)$ and $X_2 = (\sqrt{R^2 - \tilde{R}^2 R^2}, -\tilde{R} R)$, respectively, rotated around $o$ with angle $\Phi$,
- $\{X_n\}_{n=3}^{\infty}$ is distributed according to a stationary Poisson process in $\mathbb{R}^2 \setminus B(o, r)$ with intensity $\lambda$ given $R = r$.

Proof

See Baumstark & Last (2007)
Representation of $\tilde{T}$ for Poisson-Voronoi tessellations

**Theorem**

Let $T$ be a PVT with intensity $\gamma = 2\sqrt{\lambda}$ induced by a stationary Poisson process with intensity $\lambda$. Let $R^2$, $\tilde{R}^2$ and $\Phi$ be independent random variables, where

- $R^2$ **gamma distributed** with parameters $1.5$ (shape) and $1/(\lambda\pi)$ (scale),
- $\tilde{R}^2$ **beta distributed** with parameters $1$ and $1/2$,
- $\Phi$ **uniformly distributed** on $[0, 2\pi)$.

Then $\tilde{T}$ is the **Voronoi tessellation** induced by $\{X_n\}_{n=1}^{\infty}$, where

- $X_1$ and $X_2$ are given by the points $X_1 = (\sqrt{R^2 - \tilde{R}^2 R^2}, \tilde{R} R)$ and $X_2 = (\sqrt{R^2 - \tilde{R}^2 R^2}, -\tilde{R} R)$, respectively, rotated around $0$ with angle $\Phi$,
- $\{X_n\}_{n=3}^{\infty}$ is distributed according to a stationary Poisson process in $\mathbb{R}^2 \setminus B(o, r)$ with intensity $\lambda$ given $R = r$.

**Proof**  See Baumstark & Last (2007)
Typical Voronoi cell of Cox processes on PVT

Line segment through the origin with the generating points $X_1$ and $X_2$, where $R_1 = R\tilde{R}$
Typical Voronoi cell of Cox processes on PVT

- Simulate two points $X_1$ and $X_2$ (grey) generating the segment through $o$, ...
Typical Voronoi cell of Cox processes on PVT

- Simulate two points $X_1$ and $X_2$ (grey) generating the segment through $o$,
- Simulate points $X_3, X_4, \ldots$ of a stationary Poisson process in $\mathbb{R}^2 \setminus B(o, r)$ with intensity $\lambda$ given $R = r$. 
Typical Voronoi cell of Cox processes on PVT

Place points on the edges of underlying Voronoi cells and construct Initial cell.
Typical Voronoi cell of Cox processes on PVT

Intersect initial cell by bisectors
Typical Voronoi cell of Cox processes on PVT
Typical Voronoi cell of Cox processes on PVT

Stopping criterion
General representation of $\tilde{T}$ for stationary tessellations

**Theorem**

*Let $T$ be an arbitrary stationary tessellation,*

\[ E_{\tilde{T}} = E_{\nu_1}(E_{\tilde{T}} h(T_{\ast} - Z)), \]

*where the random variable $Z$ is uniformly distributed on $E_{\tilde{T}}$ given $T_{\ast}$.***
General representation of $\tilde{T}$ for stationary tessellations

**Theorem**

Let $T$ be an arbitrary stationary tessellation,

- $T^*$ the conditional version of $T$ under the Palm distribution with respect to the vertices of $T$, 

Then, for any measurable function $h: \mathbb{N} \rightarrow [0, \infty)$,

$$E[h(\tilde{T})] = E[\nu_1(E^*)] E[h(T^* - Z)],$$

where the random variable $Z$ is uniformly distributed on $E^*$ given $T^*$. 

Application to Poisson-Delaunay tessellations

- The distribution of $\tilde{T}$ can be expressed by the distribution of $(T^*, E^*)$.

- If $T$ is a PDT, then the vertices of $T$ form a stationary Poisson process and $(T^*, E^*)$ can be easily simulated using Slivnyak's theorem.
General representation of $\tilde{T}$ for stationary tessellations

Theorem

Let $T$ be an arbitrary stationary tessellation,

- $T^*$ the conditional version of $T$ under the Palm distribution with respect to the vertices of $T$,
- $E^*$ the edge star of $T^*$ at $o$, and

$$E h(\tilde{T}) = \mathbb{E} \nu_1(E^*) \mathbb{E}(\nu_1(E^*-Z)),$$

where the random variable $Z$ is uniformly distributed on $E^*$ given $T^*$. 

Application to Poisson-Delaunay tessellations

- The distribution of $\tilde{T}$ can be expressed by the distribution of $(T^*, E^*)$.
- If $T$ is a PDT, then the vertices of $T$ form a stationary Poisson process and $(T^*, E^*)$ can be easily simulated using Slivnyak’s theorem.
General representation of $\tilde{T}$ for stationary tessellations

**Theorem**

Let $T$ be an arbitrary stationary tessellation,

- $T^*$ the conditional version of $T$ under the Palm distribution with respect to the vertices of $T$,
- $E^*$ the *edge star* of $T^*$ at $o$, and
- $\tilde{T}$ the conditional version of $T$ given that $o \in T^{(1)}$.

Then, for any measurable function $h: N \rightarrow [0, \infty)$,

$$E_h(\tilde{T}) = \frac{1}{E(\nu_1(E^* - Z))} E(\nu_1(E^* h(T^* - Z)))$$

where the random variable $Z$ is uniformly distributed on $E^*$ given $T^*$.
General representation of $\tilde{T}$ for stationary tessellations

**Theorem**

Let $T$ be an arbitrary stationary tessellation,

- $T^*$ the conditional version of $T$ under the Palm distribution with respect to the vertices of $T$,
- $E^*$ the edge star of $T^*$ at $o$, and
- $\tilde{T}$ the conditional version of $T$ given that $o \in T^{(1)}$.

Then, for any measurable function $h : \mathbb{N}_F \to [0, \infty)$,

$$
\mathbb{E} h(\tilde{T}) = \frac{1}{\mathbb{E} \nu_1(E^*)} \mathbb{E} \left( \nu_1(E^*) \cdot h(T^* - Z) \right),
$$

where the random variable $Z$ is uniformly distributed on $E^*$ given $T^*$. 

---

*Application to Poisson-Delaunay tessellations*

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General representation of $\tilde{T}$ for stationary tessellations

**Theorem**

Let $T$ be an arbitrary stationary tessellation,

- $T^*$ the conditional version of $T$ under the Palm distribution with respect to the vertices of $T$,
- $E^*$ the edge star of $T^*$ at $o$, and
- $\tilde{T}$ the conditional version of $T$ given that $o \in T^{(1)}$.

Then, for any measurable function $h : \mathbb{N}_F \rightarrow [0, \infty)$,

$$
\mathbb{E} h(\tilde{T}) = \frac{1}{\mathbb{E} \nu_1(E^*)} \mathbb{E} \left( \nu_1(E^*) h(T^* - Z) \right),
$$

where the random variable $Z$ is uniformly distributed on $E^*$ given $T^*$.

**Application to Poisson-Delaunay tessellations**

- The distribution of $\tilde{T}$ can be expressed by the distribution of $(T^*, E^*)$. 
General representation of $\tilde{T}$ for stationary tessellations

Theorem

Let $T$ be an arbitrary stationary tessellation,

- $T^*$ the conditional version of $T$ under the Palm distribution with respect to the vertices of $T$,
- $E^*$ the edge star of $T^*$ at $o$, and
- $\tilde{T}$ the conditional version of $T$ given that $o \in T^{(1)}$.

Then, for any measurable function $h : \mathbb{N}_F \to [0, \infty)$,

$$\mathbb{E} h(\tilde{T}) = \frac{1}{\mathbb{E} \nu_1(E^*)} \mathbb{E} \left( \nu_1(E^*) h(T^* - Z) \right),$$

where the random variable $Z$ is uniformly distributed on $E^*$ given $T^*$.

Application to Poisson-Delaunay tessellations

- The distribution of $\tilde{T}$ can be expressed by the distribution of $(T^*, E^*)$.
- If $T$ is a PDT, then the vertices of $T$ form a stationary Poisson process.
Theorem

Let $T$ be an arbitrary stationary tessellation,

- $T^*$ the conditional version of $T$ under the Palm distribution with respect to the vertices of $T$,
- $E^*$ the edge star of $T^*$ at $o$, and
- $\tilde{T}$ the conditional version of $T$ given that $o \in T^{(1)}$.

Then, for any measurable function $h : \mathbb{N} \rightarrow [0, \infty)$,

$$
\mathbb{E}h(\tilde{T}) = \frac{1}{\mathbb{E} \nu_1(E^*)} \mathbb{E} \left( \nu_1(E^*) h(T^* - Z) \right),
$$

where the random variable $Z$ is uniformly distributed on $E^*$ given $T^*$.

Application to Poisson-Delaunay tessellations

- The distribution of $\tilde{T}$ can be expressed by the distribution of $(T^*, E^*)$.
- If $T$ is a PDT, then the vertices of $T$ form a stationary Poisson process and $(T^*, E^*)$ can be easily simulated using Slivnyak’s theorem.
Cox processes on Poisson-Delaunay tessellations

Cox process on PDT and its Voronoi tessellation
Typical Voronoi cell of Cox processes on PDT

Start: Simulate typical edge star $E^*$ using Slivnyak’s theorem
Typical Voronoi cell of Cox processes on PDT

Initial cell
Typical Voronoi cell of Cox processes on PDT

Cell cut by bisectors
Typical Voronoi cell of Cox processes on PDT

Stop: Weight cell characteristic by $\nu_1(E^*)/\mathbb{E}\nu_1(E^*) = \nu_1(E^*)/(64/(3\pi \sqrt{\lambda}))$
Cox processes on Poisson line tessellations

Cox process on PLT and its Voronoi tessellation
Typical Voronoi cell of Cox processes on PLT

Theorem

Let

1. $T^{(1)}$ the edge set of a stationary PLT of intensity $\gamma$,
2. $\ell(\Phi)$ the line with $o \in \ell(\Phi)$ and direction $\Phi \sim U[0, \pi]$ independent of $T^{(1)}$,
3. $\tilde{T}$ the conditional version of $T$ given that $o \in T^{(1)}$, 

Proof

Slivnyak's theorem

Remark:

Note that $T^{(1)} = \bigcup n \in \mathbb{Z} \ell(\Phi_n, R_n)$, where

1. $\{R_n\}$ a stationary Poisson process in $\mathbb{R}$,
2. $\{\Phi_n\}$ an i.i.d. sequence independent of $\{R_n\}$ with $\Phi_n \sim U[0, \pi]$,
3. $\ell(\Phi_n, R_n) = \{(x, y) \in \mathbb{R}^2: x \sin \Phi_n - y \cos \Phi_n = R_n\}$. 


Typical Voronoi cell of Cox processes on PLT

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Typical Voronoi cell of Cox processes on PLT

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Contents

Introduction

Point processes and Palm calculus

Random tessellations

Local simulation of typical Voronoi cells

Cox processes on random tessellations

Multiscale network modeling (Outlook to part II)
Multiscale Modeling and Simulation of Networks

Consider random tessellations with inner structure of cells
Multiscale Modeling and Simulation of Networks

Consider random tessellations with **inner structure** of cells

- Insert **random graphs** into cells (wired networks) and compute the distribution of
Multiscale Modeling and Simulation of Networks

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Multiscale Modeling and Simulation of Networks

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- Insert random graphs into cells (wired networks) and compute the distribution of
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  - number of hops to the root, etc.

- Insert full-dimensional random sets into cells (wireless networks) and compute the distribution of
  - uncovered cell area (e.g., the area where the signal-to-interference ratio is below a given threshold)
  - uncovered boundary length of cells (e.g., regions where handover of mobile users might be problematic), etc.

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  - providing a formula library of analytical (simulation-based, parametric) approximation formulas
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  - a wide spectrum of multiscale tessellation models, and
  - a wide spectrum of model parameters
Multiscale Modeling and Simulation of Networks

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Multiscale Modeling and Simulation of Networks

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Multiscale Modeling and Simulation of Networks

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Multiscale Modeling and Simulation of Networks

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Multiscale Modeling and Simulation of Networks

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Multiscale Modeling and Simulation of Networks

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Multiscale Modeling and Simulation of Networks

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Thank you for your attention!