Capacity estimates of wireless networks in Poisson shot model over uniform or fractal Cantor maps

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Wireless networks: Poisson shot models

• Classic model:
  – Wireless devices
    Poisson distributed on a map
  – A density measure \( \lambda \)

Access point network architecture

- emitters $z_1, \ldots, z_i, \ldots$ transmit independent flows toward an access point $z$.
  - Signal of emitter $i$ comes to $z$ with energy level $s(z_i)$
  - Our objective: to estimate the average capacity of the system
The capacity estimation problem

- Three estimators:
  - A shannon equivalent capacity upper bound $C_S(\lambda)$
  - The flat outage capacity as a classic $C_F(K,\lambda)$
  - The optimized outage capacity used in 4G $C_B(\lambda)$
Beyond Uniform Poisson models

• The world is fractal!
  – Self similar scaled
    • The revenge of the cauliflower
Plan of the talk

• Capacity definitions and physical models
• Infinite network maps and the dark night paradox
• Outage capacities in infinite uniform maps
• The energy field Theorem and Shannon capacity
• Fractal geometries
• Capacities in Cantor maps
• Conclusion
• Entertainment: the french army versus fractals
Capacity definitions and physical model

"I'm warning you, Perkins - your flagrant disregard for the laws of physics will not be tolerated!"
Shannon capacity model

- « Non collaborative »

Shannon capacity
- Each flow is noise for the other flows
- Shannon capacity for Gaussian noise:

\[
C_s(\lambda) = E \left[ \sum_i \log_2 \left( 1 + \frac{s(z_i)}{N + \sum_{j \neq i} s(z_j)} \right) \right] \text{ bit per Hz}
\]
Flat outage capacity model

- Classic wireless
  - Packets are all coded with same rate $R$
  - SNIR threshold $K$

$$R \leq \log_2(1 + K) \text{ bit per Hertz}$$

$$C_F(K, \lambda) = E \left[ R \sum_i P \left( \frac{s_i(z)}{N + \sum_{j \neq i} s_j(z)} \geq K \right) \right] \text{ bit per Hz}$$
Optimized outage capacity

- New generation wireless
  - Transmitter $i$ optimizes its coding rate according to SNIR statistics

$$R(z_i), K(z_i)$$

- $R(z_i) \leq \log_2 (1 + K(z_i))$

$$C_B(\lambda) = E \left[ \sum_i R(z_i) P \left( \frac{s_i(z)}{N + \sum_{j \neq i} s_j(z)} \geq K(z_i) \right) \right] \text{ bit per Hz}$$
Physical assumptions

- Signal attenuation and i.i.d. fading
  \[ s(z_i) = \frac{F_i}{\|z - z_i\|^\alpha} \quad \text{for } \alpha > 2 \]
  - The Fi are i.i.d. (e.g., Rayleigh, exponential)
    - Don’t change during packet transmission.

![Graph showing fast and slow fading](image)
Infinite network map and the dark night paradox
Infinite plan network map

- Cumulated energy
  \[ S(\lambda) = N + \sum_i s_i(z) \]

- Distribution
  \[ w(\theta, \lambda) = E[\exp(-S(\lambda)\theta)] \]
  \[ w(\theta, \lambda) = E[e^{-\theta N}] \exp(-\lambda f(\theta)) \]
  - With
    \[ f(\theta) = \iint (1 - E(e^{-\theta s(z)}))dz^2 = E[F^\gamma] \iint (1 - e^{-\theta \|z\|^\alpha})dz^2 \]
    \[ \iint (1 - e^{-\theta \|z\|^\alpha})dz^2 = \pi \Gamma(1 - \gamma) \theta^\gamma \]
    - eg Rayleigh fading \[ E[F^\gamma] = \Gamma(1 + \gamma) \]
Known results

- Shannon capacity
  - Noiseless $N = 0$
    \[ C_S(\lambda) = \frac{1}{\gamma \log 2} \]

- Flat outage capacity
  \[ C_F(\lambda, K) = \frac{\log(1 + K)}{K^\gamma} \sin(\pi \gamma) \frac{1}{\pi \gamma \log 2} \]

- Optimized outage capacity
  \[ C_B(\lambda) = ? \]

Sharp or smooth horizon drop?

Horizon sharp drop

Horizon smooth drop
The dark night paradox

- When $\alpha \to 2$, $S(\lambda) \to \infty$.
  - Thus capacities are 0 when $\alpha < 2$

$$C_S(\lambda) = -E \left[ \sum_i \log_2 \left( 1 - \frac{s(z_i)}{S(\lambda)} \right) \right] \to E \left[ \frac{1}{\log 2} \sum_i \frac{s(z_i)}{S(\lambda)} \right] = \frac{1}{\log 2}$$
Generalized Poisson shot model

- Terminals uniformly distributed in $\mathbb{R}^D$
  - $\alpha > D$, $\gamma = \frac{D}{\alpha}$

\[
f(\theta) = E[F^{\gamma}]\pi_D \Gamma(1-\gamma)\theta^\gamma
\]

\[
C(\lambda) = \frac{\alpha}{D \log 2}
\]

Outage capacities in uniform maps
Case of Rayleigh fading

• Exponential
  
  \[ P(F > x) = e^{-x} \]

  \[ p(r, K) = P\left( Fr^{-\alpha} > KS(\lambda) \right) \]

  \[ p(r, K) = \int P\left( S(\lambda) < \frac{x}{K} r^{-\alpha} \right) e^{-x} dx = \exp\left( -\lambda f (Kr^{\alpha}) \right) \]

• Exercice:

  – Proove that

  \[ \frac{C_F (\lambda, K)}{\log_2 (1 + K)} = \lambda \int \int p(\|z\|, K) d\zeta^2 = \frac{\sin(\pi \gamma)}{\pi \gamma} K^{-\gamma} \]

  – Hint:

  \[ \Gamma(1+\gamma) \Gamma(1-\gamma) = \frac{\pi \gamma}{\sin(\pi \gamma)} \]
Flat outage capacity in general fading

- \( \ln R^D \)

\[
\frac{C_F(\lambda, K)}{\log_2(1 + K)} = \int \int \lambda P(KS(\lambda) - s(\|z\|) < 0)dz^D
\]

\[
= \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \lambda f(-\theta) \exp(-\lambda f(\theta K)) \frac{d\theta}{\theta}
\]

\[
= \frac{e^{i\pi\gamma} - e^{-i\pi\gamma}}{2i\pi} \int_0^{+\infty} \lambda C \theta^{-\gamma} \exp(-\lambda C(\theta K)^\gamma) d\theta
\]

\[
= \frac{\sin(\pi\gamma)}{\pi\gamma} K^{-\gamma}
\]

- The same!
Optimized outage capacity, Rayleigh fading

• Classic local optimization

\[ \frac{\partial}{\partial K} \log(1 + K) p(r, K(r)) = 0 \quad \rightarrow \quad \frac{\partial}{\partial K} \log(1 + K) \exp \left( -\lambda (K(r) r^{-\alpha})^\gamma \right) = 0 \]

Close formula

\[ C_B(\lambda) = -\frac{\sin \pi \gamma}{\pi \gamma} \frac{1}{\gamma \log 2} \int_0^\infty \exp \left( -\frac{1}{\gamma} K^\gamma \rho(K) \right) \log(1 + K) \rho'(K) dK \]

with

\[ \rho(K) = \frac{K^{1-\gamma}}{(1 + K) \log(1 + K)} \]

via classic numerical analysis

\[ \lim_{\gamma \to 1} \frac{C_B(\lambda)}{C_S(\lambda)} = \frac{1}{e} \]
Optimized outage capacity, Rayleigh fading

\[ \frac{C_B(\lambda)}{C_S(\lambda)} \]

Horizon sharp drop
Energy field theorem and Shannon capacity

Le champ de tournesol, V. van Gogh
The energy field theorem

- Total energy received
  \[ S(\lambda) = \sum_{i} s(z_i) \]
  \[
  C_s(\lambda) = E\left( \iint \log_2 \left( 1 + \frac{s(x)}{S(\lambda)} \right) \lambda dx^2 \right)
  
  C_s(\lambda) = E\left( \iint \left( \log_2 (S(\lambda) + s(x)) - \log_2 S(\lambda) \right) \lambda dx^2 \right)
  
  C_s(\lambda) = \iint \left( E[\log_2 S(\lambda) / I \cup \{x\}] - E[\log_2 S(\lambda) / I \setminus \{x\}] \right) \lambda dx^2
  
- Differential form: the energy field theorem
  - Independent of fading, noise, etc.
    \[ C_s(\lambda) = \lambda \frac{\partial}{\partial \lambda} E(\log S(\lambda)) \]
  - Notice that capacity of node \( x \) is
    \[ \frac{\partial}{\partial \delta_x} E(\log S(\lambda)) \]
Shannon capacity proof (i.i.d. fading)

• Space contraction
  – By arbitrary factor \( a \)
  – increases energy
    \[
    S(\lambda) \rightarrow a^{-\alpha} S(\lambda)
    \]

• Equivalent to density increase
  – By factor \( a^{-1/D} \)
    \[
    S(a^{-1/D} \lambda) \equiv a^{-\alpha} S(\lambda)
    \]

\[
E(\log S(\lambda)) = \frac{\alpha}{D} \log \lambda + E(\log S(1))
\]

\[
C_s(\lambda) = \lambda \frac{\partial}{\partial \lambda} E(\log_2 S(\lambda)) = \frac{\alpha}{D \log 2}
\]
Shannon capacity: hackable formula

- Via Mellin transform

\[
E[S(\lambda)^{-s}] = \frac{1}{\Gamma(s)} \int_0^\infty E[e^{-\theta S(\lambda)}] \theta^{s-1} d\theta
\]

\[
C_s(\lambda) = \int_0^\infty \lambda f(\theta) \exp(-\lambda f(\theta)) \frac{d\theta}{\theta} \times E[e^{-\theta N}]
\]
Fractal geometries

Euclid (-300) « Elements »
Uniform Poisson models are not (always) realistic

- The world is fractal
  - Fractal cities
Fractal models

- Fractal maps
  - from fractal generators
Shannon capacity theorem extended to fractal maps

- Eg Sierpinsky triangle
- Fractal dimension
  \[ d_F = \frac{\log 3}{\log 2} \approx 1.58... \]
- If extension holds, then capacity increases on fractal map

\[ C_S(\lambda) = \frac{\alpha}{d_F \log 2} + P(\log \lambda) \]

Small periodic fluctuations of mean 0
Fractal dimension in physics

- The probability of return of a random walk on the Sierpinski triangle
  - After n steps is \( \frac{1}{n^{d_F/2}} \) (Rammal Toulouse 1983)
  - generalizes the random walk in D-lattice

- Other dimension on fractals
  - Spectral dimension
    - Power laws in state density
    - Used in percolation
Fractal (Hausdorff) dimension

- Sierpinski triangle
  - Divide unit length by 2
  - Structure is divided by 3

\[
\left( \frac{1}{2} \right)^d = \frac{1}{3}
\]

\[
\left( \frac{1}{2} \right)^D = \frac{1}{4} \implies D = 2
\]
Fractal maps

Cantor maps

\[ d_F = -2 \frac{\log 2}{\log a} < 2 \]

\[ d_F = -2 \frac{\log 4}{\log a} < 2 \]
Fractal dimension of Cantor map

\[ a^{d_F} = \frac{1}{\ell^2} \]
Poisson shot on Cantor map
Cantor maps

• Support measure $\mu_F$
  – Defined recursively (dimension 1)
    $$\mu_F = \sum_{m=0}^{\ell-1} m \frac{1-a}{\ell-1} + \mu_F \circ a$$

  – Poisson shot on Cantor map
    • Mapped from a uniform Poisson shot
      $$x = b_{-n}b_{-n+1} \ldots b_0b_1b_2 \ldots \quad b_j \in \{0, \ldots, \ell-1\}$$
      $$k(x) = (1-a) \sum_j a^j \frac{b_j}{\ell-1}$$
Cantor map

- Isometries can be used in the recursion

\[ \mu_F = \sum_{m=0}^{\ell-1} z_m + \mu_F \circ aJ_m \]
Capacities in Cantor maps
Shannon Capacity in Cantor map

• Access point at left corner, arbitrary fading
  – Contraction by factor $a$
    \[ S(\lambda) \rightarrow a^{-\alpha} S(\lambda) \]
  – Density increases
    \[ \lambda \rightarrow 4\lambda \]

\[
E(\log_2 S(4\lambda)) = -\alpha \log_2 a + E(\log_2 S(\lambda))
\]

\[
E(\log_2 S(\lambda)) = -\alpha \frac{\log_2 a}{\log 4} \log \lambda + Q(\log \lambda)
\]

\[
C_s(\lambda) = \lambda \frac{d}{d\lambda} E(\log_2 S(\lambda)) = -\alpha \frac{\log_2 a}{\log 4} + Q'(\log \lambda)
\]

Periodic of Mean zero
Periodic oscillation

- Small indeed (thanks to the *no free lunch* conjecture)
- exact analysis via Fourier transform over the hackable formula
- Amplitude of order $10^{-3}$

$a = 0.3, \alpha = 4$

P. Jacquet: Capacity of simple MIMO wireless networks in uniform or fractal maps, MASCOTS 2013
Random access points

• Access point in the fractal map in position $z$
  \[ C_S(\lambda, z) \]

  – Same contraction argument

  \[ E(C_S(\lambda, z)) = \frac{\alpha}{d_F \log 2} + P_R(\log \lambda) \]

  Small periodic fluctuations of mean 0
Shannon capacity on random access points

- Oscillation amplitude?
  - Some bounds but not tight
  - Simulations show small amplitudes
Philosophical consequence

• The fractal Poisson shot model never converges toward the uniform Poisson shot model
  – Even when the fading variance tends to infinity.
Economical consequence

- The actual capacity increases significantly on fractal Poisson shot model

\[ C_S(\lambda) \approx \frac{\alpha}{d_F \log 2} \]

\[ d_F << D \]
The fractal dark night

- Straightforward analysis gives power law distance distribution

\[ \int \mu_F d\|z\|^2 = d_F r^{d_F-1} P_1(\log r) dr \quad \int \mu_F d\|z\|^2 = r^{d_F} P_2(\log r) \]

- Therefore \[ S(\lambda) = \infty \] when \[ \alpha \leq d_F \]
The fractal dark night

– Horizon is \( \alpha = d_F \ll D \)

\[
\lim_{\alpha \to d_F} C_S(\lambda) = \frac{1}{\log 2}
\]

– Sharp horizon drop!
The flat outage capacity

- We have
  \[ f(\theta) = \iiint (1 - E[e^{-\theta s(z)}]) \mu_F dz^2 \]
  \[ = E[F^{\gamma_F}] \Gamma(1 - \gamma_F) \theta^{\gamma_F} P_3(\log \theta) \]

- With Rayleigh fading
  \[ \frac{C_F(\lambda, K)}{\log_2 (1 + K)} = \lambda \iiint \exp(-\lambda f(K r^\alpha)) \mu_F dz^2 \]
  \[ = \lambda \int_0^\infty \exp\left(-\lambda \frac{\pi \gamma_F}{\sin \pi \gamma_F} K^{\gamma_F} r^{d_F} P_3(\log K + \alpha \log r)\right) d_F r^{d_F-1} P_1(r) dr \]

  - The quantity \( K^{\gamma_F} \frac{C_F(\lambda, K)}{\log(1 + K)} \) is periodic in \( K \) and \( \lambda \)
  and is \( O(\gamma_F - 1) \) when \( \gamma_F \to 1 \)

- Smooth horizon drop.
- Should hold with general fading
Optimized outage capacity

• We prove (easy) that $C_B(\lambda)$ is periodic in $\lambda$
  – Classic contraction argument.

  – We prove (hard) that its mean value satisfies

    \[
    \lim_{\gamma_F \to 1} \frac{\overline{C_B}}{C_S(\lambda)} = \frac{1}{e}
    \]

• Sharp horizon drop, like with uniform Poisson.
Conclusion

• Shannon capacity and optimized outage capacity have sharp horizon drops
• Capacities on fractal maps are much larger than capacities on uniform maps
• Capacity estimates have small periodic oscillations
• Generalization to self-similar geometries?
Thank you!

Questions?
French Army fractal tactical organization
Fractal sets: Disposition in diamond of French Army (4 corps)
Disposition of French « Corps » in four brigades

- Brigade 1
- Brigade 2
- Brigade 3
- Brigade 4

III Regiment

X brigade1
X brigade2
X brigade3
X brigade4
Organization of French Brigades in two regiments
French batallion in four companies
The French company in four sections

section1  section2  section3  section4
The French section in four « escouades »

escouade1 escouade2 escouade3 escouade4
French escouade of 16 soldiers