Analysis of Random Processes on Regular Graphs With Large Girth

Nick Wormald
Monash University

Joint work with Carlos Hoppen

Background: independent sets in regular graphs

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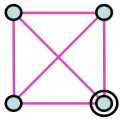

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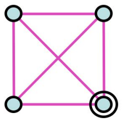

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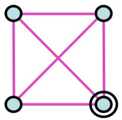

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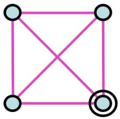

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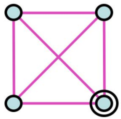

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Such results extend to graphs G with $\Delta(G) = d$.

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Bollobás: A.a.s. independent set size in the random case is at most $(2n \log d)/d$. Hence there are some large girth graphs with no independent set bigger than this.

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Random graphs are *almost* of large girth. Properties of large girth graphs often translate to the random case.

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- Remove from S the vertices incident with R. Then a.a.s. $\alpha(G) \geq c_{d,g}n - O(\log n)$.

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$$\alpha(G) \geq (\beta_1(d) - \epsilon)n$$

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Proof outline: add vertices greedily. Analyse.

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- When selecting a new vertex v, pick it randomly from those of minimum degree in the shrinking graph. ("Degree-greedy")

- Analysis [W, '95] gives a result $\beta_2(d)$ by solving differential equations. Seems always better than all previous bounds.

Even better results for random case (ctd)

d	$\beta_0(d)$	$\beta_1(d)$	$\beta_2(d)$
3	0.4139	0.3750	0.4328
4	0.3510	0.3333	0.3901
5	0.3085	0.3016	0.3566
6	0.2771	0.2764	0.3296
7	0.2528	0.2558	0.3071
8	0.2332	0.2386	0.2880
9	0.2169	0.2240	0.2716
10	0.2032	0.2113	0.2573
20	0.1297	0.1395	0.1738
50	0.0682	0.0748	0.0951
100	0.0406	0.0447	0.0572
$\to \infty$	$\to \log d/d$	$\to \log d/d$	?

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BUT these don't tell us about all large girth regular graphs.

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- D is now dominating of expected size at most $np + n(1-p)^{d+1}$.
- So some D is this small. Choose p to minimise.

  E.g. this gives $0.527n$ if $d = 3$.

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For large girth graphs

Recall β_1 from the $\mathcal{G}_{n,d}$ case:

Theorem [Lauer & W '07]

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NOTE: for $d \leq 6$, better results are known.

Similar result for maximal induced forests (Hoppen & W '08).

These arguments used expectation only.

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We can show, in a general way, bounds for all d-regular graphs of sufficiently large girth similar to the random case, with respect dominating sets, independent sets and many other objects.

The proof uses the random graph results, together with some features of their proofs.

Basic approach

We define a set of algorithms (chunky LDA). Such an algorithm $A(\epsilon)$ produces output SET satisfying the following:

(i) Sharp behaviour for random input

When $A(\epsilon)$ is applied to $\mathcal{G}_{n,d}$, a.a.s. $|\mathrm{SET}| = cn \pm \epsilon n$, $c > 0$.

(ii) Fixed expectation for large girth input

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$\{$(ii) , first moment principle$\}$ $\Longrightarrow$ all d-regular graphs of girth $\geq g(\epsilon)$ must have a SET of size $\geq cn \pm \epsilon n$. (Now let $\epsilon \to 0$.)

Important features of chunky LDAs

- We can compute the constant c and hence explicitly give their performance.
- We can approximate known powerful greedy algorithms using chunky LDAs.

Definition of LDA (Local Deletion Algorithm)

Input $G = G_0$.

For $t = 1, \ldots, N$, repeat the following Step t:

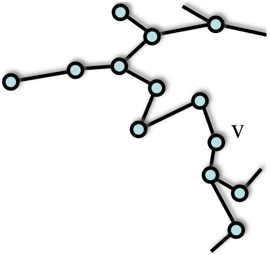
(i) Selection step : a selection rule Π_t produces a (usually, random) $S_t \subseteq G_{t-1}$ of seeds

(ii) Exploration step : for each seed $v \in S_t$, obtain a (bounded diameter) subgraph ψ_v of G_{t-1} containing v using a local rule L

(iii) Clash step : if ψ_v does not induce a tree or is adjacent to some ψ_u, all its vertices are designated as clash vertices

(iv) Insertion step : add some subset of the vertices explored (depending on the isomorphism type of the explored neighbourhood) to an output set $\mathcal{O}$. Delete all explored vertices to obtain G_t. Optionally, can colour some vertices.

Local Rule



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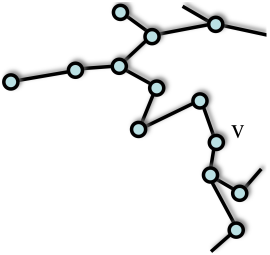

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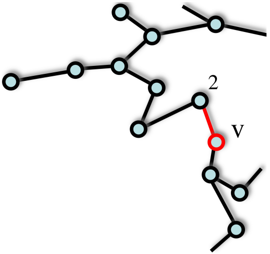

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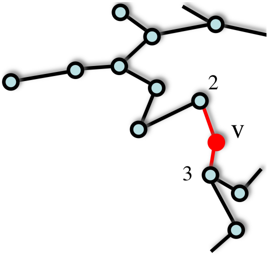

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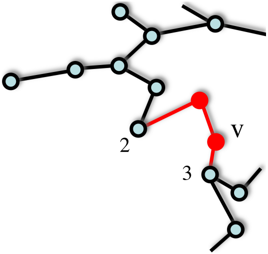

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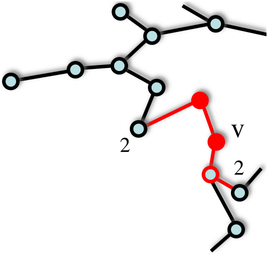

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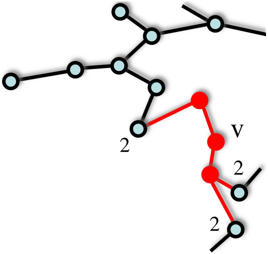

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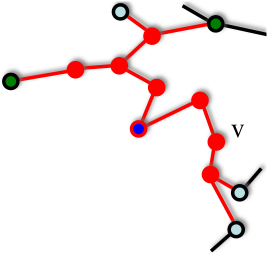

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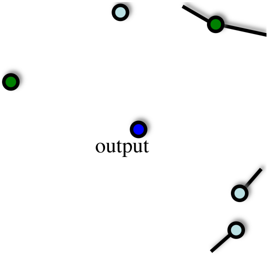

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Selection Rule, and Chunky LDA's

Example of selection rule: Pick a random vertex of degree i.

In chunky LD algorithms the selection rule consists of choosing each vertex of type j in step t as a seed independently with probability $p_{j,t}$. (Values predetermined.)

Using Chunky LDA's

For a given local rule L there is an easily described system of differential equations

$$y_j'(x) = \sum_{i=1}^{R} p_i(x) y_i(x) f_{j,i}(y_1, \ldots, y_R) \quad (1 \leq j \leq R+1)$$

such that

(i) we can construct chunky LDA's with local rule L and whose performance is described by the solutions of the d.e. system, i.e. to within accuracy ϵn, when run on a random d-regular graph.

(ii) the d.e. system also essentially describes their performance on all d-regular graphs of girth at least some $g(\epsilon)$.

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It can be deprioritised: instead of min degree, define functions $q_i(x)$ such that the probability that a degree i vertex is chosen at step t is $q_i(t/n)$.

We can choose q's to mimic the behaviour of the degree-greedy, when applied to random d-regular graphs.

These were given explicitly for a general class of prioritised algorithms (satisfying certain technical conditions) [W 03].

Those deprioritised algorithms are amenable in the following sense.

Amenable deprioritised algorithms

A local deletion algorithm is an amenable deprioritised algorithm if the selection rule is of the following type.

$\tilde{p}_i : [0, M] \to [0, 1]$ is a piecewise Lipschitz continuous function, for each possible vertex type i ($1 \le i \le R$), such that $\sum_{i=1}^{R} \tilde{p}_i(x) = 1$.

(i) A number $i \in \{1, \dots, R\}$ is chosen with probability $p_i(t, n) = \tilde{p}_i\,(t/n)$;

(ii) a vertex v of type i is chosen uniformly at random.

Chunkifying amenable deprioritised algorithms

Theorem [Hoppen & W '14+]

For each amenable deprioritised algorithm A, and $\epsilon > 0$, there is a chunky LDA that has (up to $O(\epsilon n)$) the same behaviour on regular graphs of sufficiently large girth as A does.

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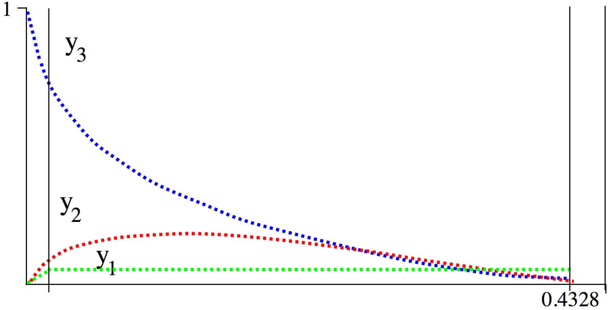

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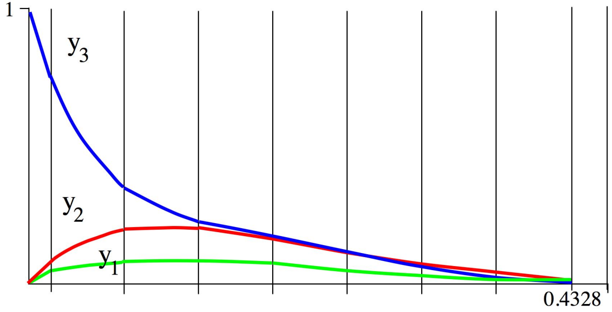

Recycled results

All the deprioritised algorithms in [W 03] are amenable. So the bounds found there for *random* d-regular graphs carry over to all d-regular graphs with sufficiently large girth.

These include the best known guarantees for the size of independent sets *, dominating sets , k-independent sets , k-dominating sets , k-independent matchings .

* For $d \geq 5$: see later.

Some other results are easily deprioritised

Some of the existing results on random d-regular graphs were obtained by degree-greedy type algorithms without deprioritising.

By creating associated amenable deprioritised algorithms, we have obtained the corresponding results for all large girth d-regular graphs in the following cases:

- Min and max bisection.
- Min connected and weakly connected dominating sets.
- Min power dominating sets in cubic graphs.

Improvements for the random case (and large girth)

A very early version of this work appeared in Hoppen's thesis ('08), obtaining the independent set result via expectation arguments only (adapting Lauer & W)

- In his thesis, Hoppen improved the results for max induced forests using a new prioritised algorithm.

Improvements for independent sets

- Kardoš, Král and Volec ('11) adapted the approach in Hoppen's thesis to obtain the lower bound $0.4352n$ for max independent set in 3-regular graphs (previously $0.4328n$).
- Improved to $0.4361n$ by Csóka, Gerencsér, Harangi and Virág using invariant Gaussian processes on the d-regular tree. Their computer simulations suggest $0.438n$ as well.
- Hoppen&W rederived the KKV result $0.4352n$ directly using the prioritised approach, and used an improved prioritised algorithm and similar analysis to get $0.4375n$ (expressed as an integral of a rational function).
- Fernholz (PhD, '07, U Texas, Austin) gives $0.43946n$ (random case only)
- Csóka (preprint '16) $0.44533n$ (better prioritised algorithm). 4-regular similar.

Improvements for max cut

- Kardoš, Král and Volec ('14) improved the lower bound for max cut in 3-regular graphs to $1.33008n$. Hoppen&W gave a short proof using an amenable deprioritised algorithm.
- Csóka (preprint '16) $1.34105n$ (again, prioritised algorithm).

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Properties of other structures (e.g. Boolean formulae) of large "girth" should follow in a similar way.

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What is the problem with chromatic number - why doesn't the random regular result carry over?

Conjecture: all 4-regular graphs of sufficiently large girth are 3-colourable.