

How much money do you (or your parents) need for retirement?

Introduction

How much money do you (or your parents) need for retirement? Some possible quick answers include:

It depends.

As much as possible.

A million dollars per year would be nice.

A billion dollars in investments should be enough.

These flippant responses miss the point, partly because the question is too poorly—and too broadly—phrased. Let's improve the question.

Different individuals will have different goals for yearly income after retirement, depending for example on their yearly income before retirement. Let's suppose, for simplicity, that an individual wishes to receive an income of $\$I$ after taxes at the start of each year for life, starting at the moment of retirement. How much would such a person require in investments at retirement in order to provide that stream of payments?

It's clear that the retiree would need I times as much as needed for a stream of $\$1$ yearly payments, so we can restrict our discussion to $\$1$ payments. The amount required in investments of course depends on the yearly rate of return after taxes on the investments, and also on whether the retiree wants the $\$1$ payments to increase over time to account for inflation. If r is the after-tax yearly rate of return and g the yearly inflation rate the retiree wishes to recognize, then a $\$1$ investment at the start of a year would grow to $\$(1 + r)$ at the end of the year. But each end-of-the-year dollar would only buy $\frac{1}{1+g}$ times as much as at the start, so the real growth factor would be

$$\frac{(1 + r)}{(1 + g)} = 1 + i, \text{ where } i = \frac{(r - g)}{(1 + g)}.$$

Let's call i the real yearly rate of return (after taxes and recognized inflation), and assume that i is non-negative.

The original question can now be phrased more precisely: "Assuming a real yearly rate of return i , how much does a retiree need to have invested in order to provide $\$1$ at the start of each year for life, starting at the moment of retirement?" And the answer: "It depends—on how long the person lives."

Future lifetimes

Let K denote the whole number of years our retiree lives after retirement. If death occurs in the first year, $K = 0$ but the retiree still will have received a total of $1 = K + 1$ payments; more generally, the retiree will always receive $K + 1$ payments whatever the value of K .

But K is unknown, of course. An extremely conservative approach would be to have enough invested to provide $\$1$ per year regardless of the size of K —that is, forever. This requires a fund of $\$\frac{1+i}{i}$: at the start of the first year the $\$1$ payment would reduce the

fund to $\frac{1+i}{i} - 1 = \frac{1}{i}$, which would grow at interest for the year to the original $(1+i)\frac{1}{i}$ at the start of the second year, allowing payments forever. If $i = 4\%$, for example, this approach would require a \$26 initial fund; making the \$1 payment would leave \$25, which at 4% interest would grow to \$26 to start the whole process again the next year.

But most people know they won't live forever and instead figure that they only need enough invested for some specific number of payments— $K + 1$ in our notation. And $K + 1$ is unknown. So we might ask: “How much is needed *on average* to provide the payments?” A common misconception is that this is the same as asking “How large a fund would be needed for *an average person* that lives *an average time* in the future?”

The average-future-lifetime mistake. Consider a simpler problem to see why it's incorrect on average to merely have enough money for the average person. There is data ([1]) that can be used to show that if you had 50,000 typical 65-year-old female retirees, if you observed the whole number of future years each lived, and if you then averaged these future lifetimes over the 50,000 retirees, your average would be highly likely to fall between 20.83 and 20.99 future years. Let's just call it 20.9 years; the corresponding average for males, by the way, would be only about 17.9 years. Throughout the remainder of this article, numerical examples will be based on such a 65-year-old female retiree, and $i = 4\%$ will be used for the rate of return. [The preceding averages result from using data appropriate for typical pensioners, as collected and analyzed by the Society of Actuaries ([1]). *Actuaries* are business people who use mathematical and statistical techniques to analyze how to provide financially now for future costs of various risks. This article is a disguised introduction to *actuarial science*, illustrating the kinds of modeling actuaries perform.]

In light of the 20.9 average future lifetime of a 65-year-old female retiree, consider an unusual retirement benefit that provides no regular yearly payments, just a single \$1,000,000 payment to any retiree that survives to age 87. Since K is on average 20.9, an average retiree would die between ages 85 and 86 and so wouldn't live to qualify for the payment at age 87. Thus \$0 is needed for an average retiree with the average future lifetime.

But if you started with 50,000 such retirees, surely some of them would live longer than the average and would collect the \$1,000,000 at age 87; had a fund started with \$ P for such a person, it would have grown with interest to $\$P(1+i)^{22}$ by that time. Thus P would have had to solve $P(1+i)^{22} = 1,000,000$ in order to make the payment, and so $P = 1,000,000v^{22}$ where v denotes $v = \frac{1}{1+i}$. That value \$ P is called the *present value* of the 22-years-later \$1,000,000. If $i = 4\%$, then $P = 421,955$.

The data used for our examples show that in fact about 25,866 of the 50,000 original retirees would be likely to survive to age 87. Thus, on average

$$\frac{(25,866)(\$1,000,000)v^{22}}{50,000}$$

would be needed in the original fund for each original retiree in order to provide the \$1,000,000 to survivors to 87. At $i = 4\%$ this is about \$218,285 per original retiree. This \$218,285 *average amount needed* is quite different from the \$0 needed for the *average retiree*.

Average amounts needed

The technique used above to analyze the average-future-lifetime mistake can be used to analyze our original pension of \$1 yearly for life. If we start with $\ell_{65} = 50,000$ 65-year-old retirees, then data would allow us to estimate the number ℓ_{65+k} of those alive k years later to receive a \$1 payment, for $k = 0, 1, 2, \dots$

The amount needed in an investment at age 65 that would grow to enough to pay \$1 to each of the ℓ_{65+k} survivors k years later is then $\ell_{65+k}v^k$, the present value of the money needed at age $65 + k$. The initial amount needed to fund *all* the payments for the lifetime of all the original retirees would then be

$$\text{\$} \sum_{k=0}^{\infty} \ell_{65+k} v^k.$$

Dividing this amount by the number ℓ_{65} of original 65-year-olds gives the average number of dollars needed per original retiree—what is called the *actuarial present value* of the payments for the lifetime of one 65-year-old:

$$APV = \sum_{k=0}^{\infty} \frac{\ell_{65+k}}{\ell_{65}} v^k.$$

For example, for the $\ell_{65} = 50,000$ female retirees discussed previously, the first few values of ℓ_{65+k} are about $\ell_{66} = 49,543$; $\ell_{67} = 49,360$; and $\ell_{68} = 48,483$. Had at most four payments been promised to survivors rather than lifelong payments, the average needed would be $\text{\$}(\ell_{65} + \ell_{66}v + \ell_{67}v^2 + \ell_{68}v^3)/\ell_{65} = \3.72 if $i = 4\%$. With lifelong payments and $i = 4\%$ the average needed turns out to be \$14.25 for that same data. Compare this with the \$26 we earlier saw is needed to guarantee \$1 payments forever rather than for life.

A probability-theory perspective

Lurking in the background of the preceding intuitive analysis are both probability and statistics, two fundamental tools for actuaries. To see this, suppose that the future lifetime X of each of a large number ℓ_0 of newborns is assumed to have the same probability distribution for each newborn. This doesn't mean that each newborn's future lifetime is the same; it just means that they all have the same chance behavior: the probability that a newborn dies in some particular age range is the same for all of the newborns. Mathematicians usually describe the random behavior in such situations by the cumulative distribution function

$$F_X(x) = \Pr[X \leq x],$$

the probability that $X \leq x$ —that is, that the newborn dies by age x . Actuaries typically describe the random behavior by the *survival function*

$$s(x) = 1 - F_X(x) = \Pr[X > x],$$

the probability that the newborn survives beyond age x . Then the expected number ℓ_x of survivors to age x from among the ℓ_0 newborns would be the fraction $s(x)$ of the original ℓ_0 newborns:

$$\ell_x = \ell_0 s(x).$$

The values ℓ_x we used intuitively earlier in this article therefore describe the distribution as well as does $F_X(x)$ since $F_X(x)$ can be computed from ℓ_x by

$$F_X(x) = 1 - s(x) = 1 - \frac{\ell_x}{\ell_0}.$$

Actuaries regularly collect statistics on large numbers of human lives in various categories in order to build models of survival functions $s(x)$ that seem appropriate for those categories. The data in [1] were prepared in this way and are represented there by a table of ℓ_x values.

Suppose now that you wanted to find the probability, denoted by actuaries by ${}_k p_{65}$, that a 65-year-old survives at least another k years. If you are willing to assume that nothing more is known about the 65-year-old other than that it is a former newborn that has survived 65 years (so there is no recent health data on the person, for instance), then this probability is just the probability that a newborn survive $65 + k$ years *given that it already survived 65 years*. That is,

$${}_k p_{65} = \frac{s(65+k)}{s(65)} = \frac{\ell_{65+k}/\ell_0}{\ell_{65}/\ell_0} = \frac{\ell_{65+k}}{\ell_{65}}.$$

Intuitively, this just says that if you divide the number of people that make it to age $65 + k$ by the number that make it to age 65, you get the fraction of 65-year-olds that survive k years—the intuitive meaning of the probability ${}_k p_{65}$.

Note that this quotient ℓ_{65+k}/ℓ_{65} , which equals ${}_k p_{65}$, appeared in the equation for APV , the actuarial present value of the \$1 payments for the lifetime of a 65-year-old. That formula for APV is therefore a sum of terms of the form ${}_k p_{65} v^k$. The factor v^k gives the present value at age 65 of \$1 at age $65 + k$. Why the factor ${}_k p_{65}$? That's the probability that the payment will in fact be made, and so it produces the *expected value* of the present value of that payment. And the APV is the sum of such terms, one for each payment that might be made. Thus, the APV is in fact the *expected value* of the *present value* of payments made so long as the retiree survives.

Of course, the *true* present value of payments made for life may well be quite different from the *expected* present value of those payments—much smaller if the retiree dies soon, and much larger if the retiree lives a long time. The true present value of the $K + 1$ payments actually made is just

$$1 + v + v^2 + \dots + v^K = \frac{1 - v^{K+1}}{1 - v}.$$

With our example data and our usual $i = 4\%$, this will be larger than the \$14.25 APV if $K + 1 \geq 21$, that is, if the retiree survives at least 20 whole years. The probability ${}_{20} p_{65} = \frac{\ell_{85}}{\ell_{65}}$ of surviving 20 years turns out to be about 0.6. This means that about 60% of the retirees that start out with the APV —the average amount needed for a lifetime of payments—would in fact run out of money before running out of life. Greater confidence in having enough money requires a greater initial fund; for example, for 99% confidence (that is, for only 1% of retirees to run out of money) the initial fund needed for an individual

can be shown to be about \$20.58—almost the \$26 needed to guarantee payments forever. In order to protect better at lower cost against running out of money, retirees often pool their risks.

Risk pooling

Although the fund needed to provide lifelong payments to an individual can vary a great deal depending on the individual’s future lifetime, in large groups of individuals these variations tend to average out: retirees that live a long time and require a large initial fund are balanced by those living a short time. Large corporate pension plans and insurance companies provide the opportunity for individuals to pool their risk and benefit from the more regular behavior of large groups.

For a large group of N 65-year-old retirees, what governs whether the total initial fund is adequate to provide lifelong payments to all the retirees is just the *sum* over all the retirees of the present value of the payments to each retiree. And mathematicians have proved that the sum of a large number of independent random variables usually is well approximated by a well-known and well-understood variable, a *normal* random variable described by the famous “bell-shaped curve”. The larger N becomes, the thinner and taller the bell becomes, indicating that values are heavily concentrated near the average.

Suppose for example that each of N retiring 65-year-old females will deposit the amount $\$P_N$ into a fund earning $i = 4\%$. How large need P_N be in order that we can be 99% confident that the total fund will be able to provide lifelong payments to all N retirees? Some retirees might use more than their $\$P_N$ deposit, and some might use less. But that’s OK so long as the total fund for the entire group covers the payments to the entire group—or at least that it would do so 99% of the time such groups of retirees were observed. It is possible to show that for large N the needed amount per person is

$$P_N = 14.25 + \frac{10.34}{\sqrt{N}},$$

which decreases to the \$14.25 *APV*—the average amount needed—as N increases. For $N = 100$, for example, $P_{100} = 15.29$, while for $N = 10,000$ it is $P_{10,000} = 14.35$. These numbers compare rather favorably with the 25.58 needed for a single individual to be 99% confident.

Generalizations

These same ideas of actuarial science—how investments grow, effects of inflation, present value, probability, statistics, actuarial present value, and so on—can be used to analyze a wide range of similar problems. For example: how much companies need contribute regularly to special funds to meet their future pension and health-care obligations to retirees; how high premiums need be for life or health or auto or homeowners insurance; how the costs of leasing equipment compare to those of buying; how a seat-belt law might influence future injury claims; how a particular new disease will impact health-care costs; and so on.

The original question

In case you still wonder how much you (or your parents) need for retirement, here's the answer: "It depends." But at least now you know more about what it depends *on*, and *how*.

Reference

1. *1994 Uninsured Pensioners Mortality Table*, Transactions of the Society of Actuaries, vol. XLVII (1995), pp.819-863.