

**Introduction to Real Analysis**  
**M361K**

## *Preface*

These notes are for the basic real analysis class. (The more advanced class is M365C.) They were written, used, revised and revised again and again over the past five years. The course has been taught 12 times by eight different instructors. Contributors to the text include both TA's and instructors: Cody Patterson, Alistair Windsor, Tim Blass, David Paige, Louiza Fouli, Cristina Caputo and Ted Odell.

The subject is calculus on the real line done the right way. The main topics are sequences, limits, continuity, the derivative and the Riemann integral. It is a challenge to choose the proper amount of preliminary material before starting with the main topics. In early editions we had too much and decided to move some things into an appendix to Chapter 2 (at the end of the notes) and to let the instructor choose what to cover. We also removed much of the topology on  $\mathbb{R}$  material from Chapter 3 and put it in an appendix. In a one semester course we are able to do most problems from Chapter 3–6 and a selection of certain preliminary problems from Chapter 2 and the two appendices.

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## CHAPTER 1

### Introduction

#### Goals

The purpose of this course is three-fold:

- (1) to provide an introduction to the basic definitions and theorems of real analysis.
- (2) to provide an introduction to writing and discovering proofs of mathematical theorems. These proofs will go beyond the mechanical proofs found in your Discrete Mathematics course.
- (3) to experience the joy of mathematics; the joy of personal discovery.

#### Proofs

Hopefully all of you have seen some proofs before. A proof is the name that mathematicians give to an explanation that leaves no doubt. The level of detail in this explanation depends on the audience for the proof. Mathematicians often skip steps in proofs and rely on the reader to fill in the missing steps. This can have the advantage of focusing the reader on the new ideas in the proof but can easily lead to frustration if the reader is unable to fill in the missing steps. More seriously these missing steps can easily conceal mistakes; most mistakes in proofs begin with “it is clear that”.

In this course we will try to avoid missing any steps in our proofs; each statement should follow from a previous one by a simple property of arithmetic, by a definition, or by a previous theorem, and this justification should be clearly stated. Writing clear proofs is a skill in itself. Often the shortest proof is not the clearest.

There is no mechanical process to produce a proof but there are some basic guidelines you should follow. The most basic is that every object that appears should be defined; when a variable, function, or set appears we should be able to look back and find a statement defining that object:

- (1) Let  $\varepsilon > 0$  be arbitrary.
- (2) Let  $f(x) = 2x + 1$ .
- (3) Let  $A = \{x \in \mathbb{R} : x^{13} - 27x^{12} + 16x^2 - 4 = 0\}$ .
- (4) By the definition of continuity there exists a  $\delta > 0$  such that...

Always watch out for hidden assumptions. In a proof, you may want to say “Let  $x \in A$  be arbitrary,” but this does not work if  $A = \emptyset$  (the empty set). A common error in real analysis is to write  $\lim_{n \rightarrow \infty} a_n$  or  $\lim_{x \rightarrow a} f(x)$  without first checking whether the limit exists (often this is the hardest part).

The audience for which a proof is intended is what determines how the proof is written. Your audience is another student in the class who is clueless as to how to prove the theorem.

### Logic

We will avoid using logical notation in our definitions and statements of theorems. Instead we will use their English equivalents. Beyond switching to the contrapositive and negating a definition, formal logical manipulation is rarely helpful in proving statements in real analysis.

You should be familiar with the basic logical operators: if  $P$  and  $Q$  are propositions, i.e., statements that are either true or false, then you should understand what is meant by

- (1) not  $P$
- (2)  $P$  or  $Q$  – the mathematical use of “or” is not exclusive so  $P$  or  $Q$  is true even if both  $P$  and  $Q$  are true.
- (3)  $P$  and  $Q$
- (4) if  $P$  then  $Q$  (or “ $P$  implies  $Q$ ”)
- (5)  $P$  if and only if  $Q$  (sometimes written  $P$  is equivalent to  $Q$ )

Similarly if  $P(x)$  is a predicate, that is a statement that becomes a proposition when an object such as a real number is inserted for  $x$ , then you should understand

- (1) for all  $x$ ,  $P(x)$  is true
- (2) there exists an  $x$  such that  $P(x)$  is true

Simple examples of such a  $P(x)$  are “ $x > 0$ ” or “ $x^2$  is an integer.”

You should know the formulae for negating the various operators and quantifiers.

Most of our theorems will have the form of implications: “if  $P$  then  $Q$ .”  $P$  is called the hypothesis and  $Q$  the conclusion.

**Definition.** The **contrapositive** of the implication “if  $P$  then  $Q$ ” is the implication “if not  $Q$  then not  $P$ .” The contrapositive is logically equivalent to the original implication. This means that one is valid (true) if and only if the other is valid.

Sometimes it is much easier to pass to the **contrapositive** formulation when proving a theorem.

**Definition.** The **converse** of the implication if  $P$  then  $Q$  is the implication if  $Q$  then  $P$ . The converse is **not** logically equivalent to the original implication.

**Definition.** A statement that is always true is called a **tautology**. An implication that is a tautology is called a **valid argument**. A statement that is always false is called a **contradiction**.

To show that an argument is not valid it suffices to find one situation in which the hypotheses are true but the conclusion is false. Such a situation is called a **counterexample**.

One technique of proof is by contradiction. To prove “ $P$  implies  $Q$ ” we might assume that  $P$  is true and  $Q$  is false and obtain a contradiction. This is really proving the contrapositive.



## CHAPTER 2

### Preliminaries: Numbers and Functions

What exactly is a number?

If you think about it, to give a precise answer to this question is surprisingly difficult. As is often the case, the word ‘number’ reflects a concept of which we have some intuitive understanding, but no concrete definition. We will try to describe exactly what we should expect from a number system. These expectations will lead us to conclude that a number is an element of  $\mathbb{R}$ , the collection of *real numbers*. Most of us have probably heard of the real numbers, but may not be exactly sure what they are.

In particular, we might ask: Why exactly are the real numbers so important? Is there some other system that would also suffice to be our system of numbers? What is wrong with just using the natural numbers, the integers, or the rational numbers? To answer this question, we will have to pin down the expectations we have on a number system. More precisely, we will have to decide exactly what properties should hold in a system of numbers. It is actually not so difficult to make progress in this area: it only requires a little bit of introspection and memory.

Think back to when you first met the idea of a number. Probably the very first purpose of a number in your life is that it allowed you to count things: 50 states, 32 professional football teams, 7 continents, 5 golden rings, etc. Needing to count things leads us to the invention (or discovery depending on your point of view) of the natural numbers (the numbers  $1, 2, 3, 4, 5, \dots$ ). Mathematicians typically denote the collection of natural numbers by the symbol ‘ $\mathbb{N}$ .’ Though this collection can be constructed quite rigorously from the standard axioms of mathematics, we will assume that we are all familiar with the natural numbers and their basic properties (such as the concept of mathematical induction; see appendix to Chapter 2). The natural numbers fulfill quite successfully our goal of being able to count.

The next thing that will expect of our number system is that it should be able to answer questions like the following: “If the Big Twelve has 12 football teams and the Big Ten has 11 (shockingly it’s true), how many teams do the conferences have between them?” In other words we will need to add. We will also multiply. The natural numbers are already well-suited for these tasks. Really this should not come as a surprise. After all, adding is really just a different way of looking at counting (i.e.,

adding three and five is the same as taking three dogs and five cats and counting the total number of animals). As we all know, multiplication is really repeated addition.

Having addition naturally leads us to subtraction. This is the first place the natural numbers will fail us. Subtracting 7 from 2 is an operation that cannot be performed within  $\mathbb{N}$ . The need for subtraction, therefore, is one of the reasons that  $\mathbb{N}$  will not work as our entire number system. Thus we are led to expand to the integers. As we all probably know, the integers are comprised of the natural numbers, the number zero, and the negatives of the natural numbers (at this point, you might protest and say that zero should be included as a natural number as it allows us to count collections which contain no objects; in fact many mathematicians do include zero in  $\mathbb{N}$ , but the distinction is of little importance). The collection of integers is denoted by  $\mathbb{Z}$ . Again we will assume we know all the basic properties of  $\mathbb{Z}$  (like prime factorization).

The integers are a very good number system for most purposes, but they still have an obvious defect: we cannot divide. Surely any reasonable number system allows division. If you and I have a sandwich and we each want an equal share, how much do we each get? Needing division, we throw in fractions: symbols which are comprised of two integers, one in the numerator and one in the denominator (of course the denominator is not allowed to be zero). A fraction will represent the number which results when the numerator is divided by the denominator. Things start to get a little bit complicated here. We can now have more than one symbol that stand for the same number:  $\frac{2}{2}$  and  $\frac{1}{1}$  will both stand for integer 1.

Combining all the numbers we have so far gives  $\mathbb{Q}$ , the collection of rational numbers. Again, we will assume that we are familiar with all its basic properties. Before we go on to justify our assertion that  $\mathbb{Q}$  is not a sufficient number system, we have another property to point out. Notice that most of our properties so far involve *operations* among our numbers: namely addition, subtraction, multiplication, and division. We call these types of properties *algebraic* (In mathematics, the word *algebra* describes the study of operations). The property we are going to discuss next is not algebraic.

Suppose then that I pick a rational number and you pick another. We can easily decide which is bigger: Namely  $\frac{a}{b}$  is bigger than  $\frac{c}{d}$  (where  $a$ ,  $b$ ,  $c$ , and  $d$  are integers) if  $ad$  is bigger than  $bc$  (assuming  $b$  and  $d$  both positive; we can easily assure both denominators are positive by moving any negative into the numerator). Since  $ad$  and  $bc$  are integers, we know how to compare them (because we know how to compare natural numbers and how to take negatives into account). Since we can always compare any two rational numbers in this way, we say that  $\mathbb{Q}$  is totally *ordered*.

In retrospect, we should have demanded this property of our number system from the beginning. Numbers should come with some notion of size. Fortunately, we got it for free. Moreover, it is interesting to notice that our expectation that a number system should include the natural numbers and that it should have certain algebraic properties is enough to lead us to include all of  $\mathbb{Q}$ . We did not request that our

system be ordered to find  $\mathbb{Q}$ . The order properties turn out to be more important in telling us which potential numbers we should NOT include (such as the imaginary number  $i$ ).

$\mathbb{Q}$  comes very close to satisfying everything we want in a number system. Unfortunately it is still lacking. Suppose we draw a circle whose diameter is 1. The length around the circle (usually called the circumference) should certainly be a number. If, however, we restrict ourselves to the rational numbers, this length will not be a number (the number is of course usually denoted  $\pi$  and it is not a rational number). The same could be said of the length of one of the sides of a square whose area is 2 (this number is usually denoted  $\sqrt{2}$ ).

These two examples merely comprise our attempt to give (geometric) demonstration that  $\mathbb{Q}$  is lacking as a number system. The real (more general) property that we seek, called *completeness*, is actually quite subtle and has to do with the presence of something like ‘gaps’ in  $\mathbb{Q}$  (the absence of the number  $\sqrt{2}$  or the number  $\pi$  is an example of such a gap). These gaps have to do with something called a *Cauchy sequence* which we will study in detail in this course. One consequence of filling in these gaps is that we are able to perform calculus. This, in turn, allows us to express all the lengths (and areas, volumes, etc.) of geometric objects like the examples above as real numbers.

One of the fundamental results in mathematics is that the collection of real numbers is the ONLY system of numbers which satisfies all of our demands. We will formulate our demands precisely throughout the first three sections of this chapter with the exception of the completeness axiom). We are thus forced to admit that the real numbers comprise the only possible choice of a number system. To give an exact definition of the real number is surprisingly complicated. In fact, the first rigorous construction of the real numbers was given by Georg Cantor as late as 1873 (by comparison, the rational numbers were constructed in ancient times).

For our purposes, we will take it on faith that the real numbers exist as a number system. In the appendix to this chapter, we will state precisely the important properties of the real numbers (which we will also take on faith) and note that these line up with our expectations of the number system (except for the completeness axiom whose importance might not be clear at first; keep our geometric example in mind). In the next chapter, we will also describe a way to define the real numbers rigorously using decimal expansions (there are actually several well-known ways).

Finally, it is important to realize that the properties given in the appendix (which we will call *axioms*), together with the completeness axiom, are the ONLY properties that we assume about  $\mathbb{R}$ . Strictly speaking, any other statement we want to make must be proven from either from our axioms or from properties we have already assumed about  $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{Q}$  (or, of course, some combination of the two).

In general, however, this can get to be a little bit tedious. Hence we will allow you to assume all of the ‘basic’ or ‘obvious’ properties of the real numbers. Unfortunately,

deciding which properties are obvious is a subjective process. Therefore, if there is any doubt about whether a statement is obvious, you should prove it rigorously from the axioms (or at least describe how to prove it rigorously). Actually, the ability to decide when statements are obvious or ‘trivial’ is an important skill in mathematics. Possessing this ability can often be a reflection of great mathematical maturity and insight.

In the appendix to this chapter, we will also derive some properties of  $\mathbb{R}$  that follow from our axioms. We will work on some of these in class, but thereafter you may consider them “known.” The appendix also contains a discussion of basic set theory and induction.

### 1. The Absolute Value

An important property of the real numbers is that we can define a size on them, which we call the absolute value. The definition is relatively simple and yet it has vast and important consequences.

**Definition.** Given a real number  $a \in \mathbb{R}$ , we define the absolute value of  $a$ , denoted  $|a|$  to be  $a$  if  $a$  is nonnegative and  $-a$  if  $a$  is negative.

**2.1.** For  $a \in \mathbb{R}$ : 1)  $|a| \geq 0$ , 2)  $|a| = 0$  if and only if  $a = 0$ , and 3)  $|a| \geq a$ .

The following technical observations will be of assistance in some arguments involving the absolute value.

**2.2.** For all  $a \in \mathbb{R}$ ,  $a^2 = |a|^2$ .

**2.3.** For all  $a, b \in \mathbb{R}$  with  $a, b \geq 0$  we have  $a^2 \leq b^2$  if and only if  $a \leq b$ . Likewise,  $a < b$  if and only if  $a^2 < b^2$ .

The next statement gives two fundamental properties of the absolute value.

**2.4.** Let  $a, b \in \mathbb{R}$ , then

- (1)  $|ab| = |a||b|$  and
- (2)  $|a + b| \leq |a| + |b|$

**Hint:** One can prove these by laboriously checking all the cases (e.g.,  $a > 0$ ,  $b \leq 0$ ) but in each case an elegant proof is obtained by using our observations to eliminate the absolute value and then proceeding using the properties of arithmetic.

The second inequality above is perhaps the most important inequality in all of analysis. It is called the **triangle inequality**.

The remaining results in this section are important consequences of the triangle inequality.

**2.5.** Let  $a, b, c \in \mathbb{R}$ . Then we have

- (1)  $|a - b| \geq ||a| - |b||$  and  
 (2)  $|a - c| \leq |a - b| + |b - c|$ .

You have probably measured the length of something using a yardstick. You might place one end at zero and see where the other end lies to get the length. In a tight place you might place the yardstick and find one end at 7" and other at 13" and conclude the length is  $13'' - 7'' = 6''$ . If I told you one end was at  $x$  and the other was at  $y$ , what would be the length? Well,  $y - x$  if  $y > x$  and  $x - y$  if  $x > y$ . In short, it would be  $|x - y|$ . So, we can regard  $|x - y|$  as the **distance** between the numbers  $x$  and  $y$ . Note this works for all real numbers, even if one or both is negative. Note that this explains why (see 2.5(2)) we call 2.4(2) the triangle inequality.

2.5(1) is often called the **reverse triangle inequality**.

**2.6.** Let  $x, \varepsilon \in \mathbb{R}$  with  $\varepsilon > 0$ . Then we have the following two statements.

- (1)  $|x| \leq \varepsilon$  if and only if  $-\varepsilon \leq x \leq \varepsilon$ . The double inequality  $-\varepsilon \leq x \leq \varepsilon$  means  $-\varepsilon \leq x$  and  $x \leq \varepsilon$ .  
 (2) If  $a \in \mathbb{R}$ ,  $|x - a| \leq \varepsilon$  if and only if  $a - \varepsilon \leq x \leq a + \varepsilon$ .

**Remark.** By a similar proof, the same properties hold with  $\leq$  replaced by  $<$ .

## 2. Intervals

Intervals are a very important type of subset of  $\mathbb{R}$ . Loosely speaking they are sets which consist of all the numbers between two fixed numbers, called the endpoints. We also (informally) allow the endpoints to be  $\pm\infty$ . Depending on whether the endpoints are finite and whether we include them in our sets, we arrive at 9 different types of intervals in  $\mathbb{R}$ .

**Definition.** An interval is a set which falls into one of the following 9 categories (assume  $a, b \in \mathbb{R}$  with  $a < b$ ). We apply the word 'bounded' if both the endpoints are finite. Otherwise we use the word 'unbounded'.

- (1) Bounded open intervals are sets of the form

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$

- (2) Bounded closed interval are sets of the form

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

- (3) There are two type of half-open bounded intervals. One type is sets of the form

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}.$$

- (4) The other is sets of the form

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}.$$

- (5) There are also two types of unbounded open intervals not equal to  $\mathbb{R}$ . One type is sets of the form

$$(a, +\infty) := \{x \in \mathbb{R} : a < x\}.$$

- (6) The other is sets of the form

$$(-\infty, b) := \{x \in \mathbb{R} : x < b\}.$$

- (7) There are two types of unbounded closed intervals not equal to  $\mathbb{R}$ . One type is sets of the form

$$[a, +\infty) := \{x \in \mathbb{R} : a \leq x\}.$$

- (8) The other is sets of the form

$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}.$$

- (9) The whole real line  $\mathbb{R} = (-\infty, \infty)$  is an interval. We count  $\mathbb{R}$  as being open, closed, and unbounded.

Some authors include the empty set,  $\emptyset$ , and single points,  $\{a\}$  for some  $a \in \mathbb{R}$ , as intervals. To distinguish these special sets, people often call them ‘degenerate intervals’ whereas sets of the above would be ‘nondegenerate intervals.’ We will reserve the word ‘interval’ for the nondegenerate case. That is, in our language, an interval is not allowed to be  $\emptyset$  or  $\{a\}$ .

**Definition.** The closure of an interval  $I$ , denoted  $\bar{I}$ , is the union of  $I$  and its endpoints.

Thus

$$\begin{aligned} \overline{(a, b)} &= \overline{[a, b]} = \overline{[a, b]} = \overline{(a, b)} = [a, b] \\ \overline{(a, +\infty)} &= \overline{[a, +\infty)} = [a, +\infty) \\ \overline{(-\infty, b)} &= \overline{(-\infty, b]} = (-\infty, b] \\ \overline{\mathbb{R}} &= \mathbb{R} \end{aligned}$$

**Definition.** The interior of an interval  $I$ , denoted  $I^\circ$ , is  $I$  minus its endpoints.

Thus

$$\begin{aligned} (a, b)^\circ &= [a, b]^\circ = [a, b]^\circ = (a, b]^\circ = (a, b) \\ (a, +\infty)^\circ &= [a, +\infty)^\circ = (a, +\infty) \\ (-\infty, b)^\circ &= (-\infty, b]^\circ = (-\infty, b) \\ \mathbb{R}^\circ &= \mathbb{R} \end{aligned}$$

### 3. Functions

Even more basic to mathematics than the concept of a number is the concept of a function. Roughly speaking, a function  $f$  from a set  $A$  to a set  $B$  is a rule that assigns

to each  $x \in A$  an element  $f(x) \in B$ . In this case we write  $f : A \rightarrow B$ . What do we mean by “rule”? Let’s try to be more precise.

**Definition.** A **function**  $f$  from  $A$  to  $B$ , denoted by  $f : A \rightarrow B$ , is a subset  $f$  of the Cartesian product  $A \times B = \{(a, b) : a \in A, b \in B\}$  satisfying

- (1) for each  $a \in A$  there exists  $b \in B$  such that  $(a, b) \in f$
- (2) for all  $a \in A$  and for all  $b, b' \in B$  if  $(a, b) \in f$  and  $(a, b') \in f$  then  $b = b'$ .

We could combine the two hypotheses into a single statement:

for each  $a \in A$  there exists a unique  $b \in B$  such that  $(a, b) \in f$

Rather than writing  $(a, b) \in f$  it is customary to write  $f(a) = b$ .

Technically then a function from  $A$  to  $B$  is just a special subset of  $A \times B$ . Mathematicians, however, rarely think of functions in this way. The idea of a rule is intuitively more accurate whereas the formal definition is just a way to make it precise.

It is very important to realize that a function is not the same thing as a formula. Many beginning students in mathematics think that in order to find a function, they need to find a formula using variables. This is not the case. It is perfectly reasonable to define a function by saying something like:

Define a function from the set of real numbers to the set  $\{0, 1\}$  by assigning the value 1 to all rational numbers and the value 0 to all irrational numbers.

Since every number has been given a value and no number has been given more than one, our rule gives a function.

This function  $f : \mathbb{R} \rightarrow \{0, 1\}$  would probably be more commonly described by saying that for  $x \in \mathbb{R}$ ,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases},$$

but either description would work. Notice that it would be impossible to find what most people would call a ‘formula’ to describe this function.

**Definition.** Let  $f : A \rightarrow B$  be a function. The set  $A$  is called the **domain** of  $f$  and  $B$  is called the **co-domain**.

The **range** of  $f$ , denoted by  $f(A)$ , is

$$f(A) = \{f(a) : a \in A\}.$$

Most of the functions we consider will be of the form  $f : \mathbb{R} \rightarrow \mathbb{R}$  or  $f : A \rightarrow \mathbb{R}$  where  $A \subseteq \mathbb{R}$ .

A function is sometimes called a **mapping** or a **transformation**. If  $f(a) = b$  we might say “ $f$  maps  $a$  to  $b$ ” or “ $f$  sends  $a$  to  $b$ .” Some functions have certain specific properties that we shall name.

**Definition.** Let  $f : A \rightarrow B$ .

- (1)  $f$  is **onto** (or **surjective**) if  $f(A) = B$ . That is,  $f$  is onto if, for every  $b \in B$ , there exists some  $a \in A$  such that  $f(a) = b$ .
- (2)  $f$  is 1–1 (or **one-to-one** or **injective**) if for all  $a_1, a_2 \in A$ ,  $f(a_1) = f(a_2)$  implies  $a_1 = a_2$ . That is,  $f$  is 1–1 if, for all  $a_1, a_2 \in A$  such that  $a_1 \neq a_2$ , we have  $f(a_1) \neq f(a_2)$ .
- (3)  $f$  is a **bijection** (or **1–1 correspondence**) if it is 1–1 and onto. This is equivalent to: for all  $b \in B$  there exists a unique  $a \in A$  with  $f(a) = b$ . (Note: it is usually simpler to show that a function is a bijection by showing it is 1–1 and onto separately.)

If a function is a bijection then you can “reverse it” to obtain a function going the other way. The following theorem makes this precise.

**Theorem 2.7.** Let  $f : A \rightarrow B$  be a bijection. Then there exists a bijection  $g : B \rightarrow A$  satisfying

- (1) for all  $a \in A$ ,  $g(f(a)) = a$ .
- (2) for all  $b \in B$ ,  $f(g(b)) = b$ .

Furthermore, this function  $g$  is unique; if  $g_1$  and  $g_2$  are bijections satisfying (1) and (2) then  $g_1 = g_2$ . (Would it be enough to assume  $g_1$  and  $g_2$  both satisfy (1)?)

The bijection  $g$  is called the **inverse function** of  $f$  and is usually denoted by  $f^{-1}$ . Do not confuse this with “ $1/f$ ”.

**Definition.** Let  $f : A \rightarrow B$ . Let  $D \subseteq A$ , and  $C \subseteq B$ .

- (1) The **image** of  $D$  under  $f$ , denoted  $f(D)$ , is

$$f(D) = \{f(x) : x \in D\}.$$

- (2) The **pre-image**, of  $C$  under  $f$ , denoted  $f^{-1}(C)$ , is

$$f^{-1}(C) = \{a \in A : f(a) \in C\}.$$

The set  $f^{-1}(C)$  always exists even when the function  $f^{-1}$  does not exist. The context of a problem will tell you which “ $f^{-1}$ ” is being used.

We are using the symbol  $f^{-1}$  in two different ways. When the function  $f^{-1}$  does exist then there are two different ways of reading  $f^{-1}(C)$ ; it can be read as the direct image of the set  $C$  under the function  $f^{-1}$  or it can be read as the inverse image of  $C$  under the function  $f$ . These give the same set and thus there is no ambiguity, in this case.

**Definition.** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . The **composition**  $g \circ f : A \rightarrow C$  is defined by  $(g \circ f)(a) = g(f(a))$ .

**2.8.** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two bijections. Then  $g \circ f : A \rightarrow C$  is also a bijection.

**Caution:** If  $f$  and  $g$  are functions, it is *not* true in general that  $f \circ g = g \circ f$ . In fact, these two compositions may have completely different domains and co-domains!

**Note:** If  $f : A \rightarrow B$  is a bijection then  $f^{-1} \circ f : A \rightarrow A$  is the identity map on  $A$  and  $f \circ f^{-1} : B \rightarrow B$  is the identity map on  $B$ . (The identity map on the set  $S$  is the map  $id : S \rightarrow S$  defined by  $id(x) = x$  for all  $x \in S$ .)



## CHAPTER 3

# Sequences

### 1. Limits

Our basic object for investigating real analysis will be the sequence.

**Definition.** A **sequence** in  $\mathbb{R}$  is a function  $f : \mathbb{N} \rightarrow \mathbb{R}$ .

**Example.**  $f(n) = n^2$  and  $g(n) = \sqrt{n}$  both define sequences in  $\mathbb{R}$ .

We typically do not use functional notation to discuss sequences, instead we write things like  $(n^2)_{n=1}^{\infty}$  or  $(\sqrt{n})_{n=1}^{\infty}$ . If we say “Consider the sequence  $(a_n)_{n=1}^{\infty}$ ,” we are referring to the sequence whose value at  $n$  is  $a_n$ . Sequences, like general functions, can be defined without using an explicit formula.

**Example.** We can define a sequence  $(a_n)_{n=1}^{\infty}$  by

$$a_n = n^{\text{th}} \text{ digit to right of the decimal point for } \pi.$$

Thus  $(a_n)_{n=1}^{\infty} = (1, 4, 1, 5, 9, \dots)$ .

**Example.** We can define a sequence recursively; that is using the previous members of the sequence to define the next element. A famous example of a recursively defined sequence is the **Fibonacci sequence** given by

$$a_1 = 1 \quad a_2 = 1 \quad a_{n+2} = a_{n+1} + a_n \text{ for } n \geq 1.$$

Find  $a_3, a_4, a_5$ .

A sequence is not to be confused with a set. For example  $\{1, 1, 1, \dots\} = \{1\}$  but  $(1, 1, 1, \dots)$  denotes  $f : \mathbb{N} \rightarrow \mathbb{R}$  with  $f(n) = 1$  for all  $n$ . This is an example of a constant sequence. A sequence is really an infinite **ordered** list of real numbers, which may repeat.

**Remark.** Sometimes we will write “consider the sequence  $(a_n)_{n=4}^{\infty}$ .” Of course this is given by  $f(n) = a_{3+n}$  for  $n \in \mathbb{N}$  so it is a sequence even though for convenience or whatever reason we chose to “start the sequence at  $n = 4$ .”

Intuitively a sequence  $(a_n)_{n=1}^{\infty}$  converges to a limit  $L$  if the terms get closer and closer to  $L$  as  $n$  gets larger and larger.

Now we need to make this idea precise so we can use it in our future discussions. First note that  $(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \dots)$  should converge to 0 under our yet to be formulated

definition. But it is not true that each term is closer to 0 than the previous term. So the rough intuitive “definition” needs clarification.

The following definition makes this intuition precise:

**Definition.** A sequence  $(a_n)_{n=1}^{\infty}$  is said to **converge** to  $L \in \mathbb{R}$  if for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  so that for all  $n \geq N$ ,  $|a_n - L| < \varepsilon$ .  $L$  is called a limit of the sequence  $(a_n)_{n=1}^{\infty}$ . If there exists an  $L \in \mathbb{R}$  such that  $(a_n)_{n=1}^{\infty}$  converges to  $L$  then we say  $(a_n)_{n=1}^{\infty}$  converges or  $(a_n)_{n=1}^{\infty}$  is a convergent sequence.

**Note:** It is important to realize that the  $N$  that you choose will typically depend on  $\varepsilon$ . Typically we expect that  $N$  will get larger as  $\varepsilon$  gets closer to 0.

This definition seems to have been first published by Bernard Bolzano, a Czech mathematician, in 1816. It is the notion of limit that distinguishes analysis/calculus from, say, algebra. This definition came about 150 years after the creation of calculus (due independently to Newton and Leibniz). The fact that it will probably take you some time to understand and become happy with it is therefore no surprise. The old guys were pretty sharp and still struggled with the notion.

**Lemma 3.1.** Let  $a \geq 0$  be a real number. Prove that if for every  $\varepsilon > 0$  we have that  $a < \varepsilon$  then  $a = 0$ .

**3.2.** Prove that if  $(a_n)_{n=1}^{\infty}$  converges to  $L \in \mathbb{R}$  and  $(a_n)_{n=1}^{\infty}$  converges to  $M \in \mathbb{R}$  then  $L = M$ . (**Hint:** argue by contradiction.)

We have just shown if a sequence has a limit then that limit is unique. Thus it makes sense to talk about *the* limit of a sequence and to write  $\lim_{n \rightarrow \infty} a_n = L$ .

**Caution:** Before we write  $\lim_{n \rightarrow \infty} a_n$  we *must* know that  $a_n$  has a limit.

**3.3.** Negate the definition of  $\lim_{n \rightarrow \infty} a_n = L$  to give an explicit definition of “ $(a_n)_{n=1}^{\infty}$  does not converge to  $L$ .”

**Remark.** We can write  $(a_n)_{n=1}^{\infty}$  does not converge to  $L$  as  $(a_n)_{n=1}^{\infty} \not\rightarrow L$ .

**Caution:** We should not write  $\lim_{n \rightarrow \infty} a_n \neq L$  to mean  $(a_n)_{n=1}^{\infty}$  does not converge to  $L$  unless we know that  $\lim_{n \rightarrow \infty} a_n$  exists.

**Definition.** A sequence  $(a_n)$  is said to **diverge** if it does not converge to  $L$  for any  $L \in \mathbb{R}$ . We will distinguish two special types of divergence:

- (1) A sequence  $(a_n)$  is said to **diverge to**  $+\infty$  if for all  $M \in \mathbb{R}$  there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$  we have  $a_n \geq M$ . In an abuse of notation we often write  $\lim_{n \rightarrow \infty} a_n = +\infty$ .
- (2) A sequence  $(a_n)$  is said to **diverge to**  $-\infty$  if for all  $M \in \mathbb{R}$  there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$  we have  $a_n \leq M$ . In an abuse of notation we often write  $\lim_{n \rightarrow \infty} a_n = -\infty$ .

**3.4.** Prove that a sequence  $(a_n)_{n=1}^{\infty}$  converges to  $L \in \mathbb{R}$  if and only if for all  $\varepsilon > 0$  the set

$$\{n \in \mathbb{N} : |a_n - L| \geq \varepsilon\}$$

is finite. This is saying that the number of terms  $a_n$  which are more than  $\varepsilon$  distance from  $L$  must be finite.

This has a couple of interesting corollaries;

- (1) if we change finitely many terms of a sequence we do not alter its limiting behaviour; if the sequence originally converged to  $L$  then the altered sequence still converges to  $L$ , and if the original sequence diverged then so does the altered sequence.
- (2) if we remove a finite number of terms from a sequence then we do not alter its limiting behaviour; if the sequence originally converged to  $L$  then the altered sequence still converges to  $L$ , and if the original sequence diverged then so does the altered sequence.

**3.5.** Prove using the definition of a limit that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

Let's examine the proof here. Your proof should look something like this.

**PROOF.** We will show  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . Let  $\varepsilon > 0$  be arbitrary but fixed. We must find  $N \in \mathbb{N}$  so that if  $n \geq N$  then  $|\frac{1}{n} - 0| < \varepsilon$  which is the same as  $\frac{1}{n} < \varepsilon$ . Choose  $N \in \mathbb{N}$  so that  $\frac{1}{N} < \varepsilon$ . Then if  $n \geq N$ ,  $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$ .  $\square$

We have used here two things. First the basic properties of order which we will assume you know. Secondly, we have used that: given  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  with  $\frac{1}{N} < \varepsilon$  or in other form,  $\frac{1}{\varepsilon} < N$ . This is called the Archimedean property.

## 2. Archimedean Property

$\mathbb{R}$  has the **Archimedean Property**, which says that for every  $x$  there exists an  $n \in \mathbb{N}$  with  $x < n$ . This seemingly innocuous property has many consequences:

- (1) For every  $\varepsilon > 0$  there exists an  $n \in \mathbb{N}$  such that

$$0 < \frac{1}{n} < \varepsilon,$$

- (2) for every  $x \in \mathbb{R}$  there exists an  $m \in \mathbb{Z}$  such that

$$m \leq x < m + 1,$$

- (3) for every  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$  there exists an  $m \in \mathbb{Z}$  such that

$$\frac{m}{n} \leq x < \frac{m+1}{n},$$

**3.6.** Using these we can prove that every real number can be approximated arbitrarily well by a rational number. We say that  $\mathbb{Q}$  is **dense** in  $\mathbb{R}$ .

(1) Prove that for every  $x \in \mathbb{R}$  and every  $\varepsilon > 0$  there exists  $\frac{m}{n} \in \mathbb{Q}$  such that

$$0 \leq x - \frac{m}{n} < \varepsilon.$$

(2) Prove that for every  $x \in \mathbb{R}$  and every  $\varepsilon > 0$  there exists  $\frac{m}{n} \in \mathbb{Q}$  such that

$$-\varepsilon < x - \frac{m}{n} \leq 0$$

Similarly, the irrational numbers are dense in  $\mathbb{R}$ .

**3.7.** Prove:

(1) For all  $\frac{m}{n} \in \mathbb{Q} \setminus \{0\}$  the number  $\sqrt{2} \frac{m}{n}$  is irrational.

(2) Prove that for every  $x \in \mathbb{R}$  and every  $\varepsilon > 0$  there exists  $\frac{m}{n} \in \mathbb{Q}$  such that

$$0 \leq x - \sqrt{2} \frac{m}{n} < \varepsilon.$$

(3) Prove that for every  $x \in \mathbb{R}$  and every  $\varepsilon > 0$  there exists  $\frac{m}{n} \in \mathbb{Q}$  such that

$$-\varepsilon < x - \sqrt{2} \frac{m}{n} \leq 0$$

### 3. Convergence

**3.8.** Prove using the definition of a limit that a)  $\lim_{n \rightarrow \infty} 1 - \frac{1}{n^2+1} = 1$ , b)  $\lim_{n \rightarrow \infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$ .

**3.9.** Prove or disprove: To disprove you need only give a counterexample.

(1) If  $\lim_{n \rightarrow \infty} a_n = L$  then  $\lim_{n \rightarrow \infty} |a_n| = |L|$ .

(2) If  $\lim_{n \rightarrow \infty} |a_n| = |L|$  then  $\lim_{n \rightarrow \infty} a_n = L$ .

(3) If  $\lim_{n \rightarrow \infty} |a_n| = 0$  then  $\lim_{n \rightarrow \infty} a_n = 0$ .

(4) If  $a_n \leq c_n \leq b_n$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = L$  then  $\lim_{n \rightarrow \infty} c_n = L$ .

This is normally referred to as the Squeeze Theorem for sequences.

**Definition.** A sequence  $(a_n)_{n=1}^{\infty}$  is called **bounded** if the set  $\{a_n : n \in \mathbb{N}\}$  is contained in a bounded interval.

**Note:** Thus  $(a_n)$  is bounded if and only if there exists  $K \geq 0$  with  $|a_n| \leq K$  for all  $n \in \mathbb{N}$ .

**3.10.** Prove that if  $(a_n)_{n=1}^{\infty}$  is a convergent sequence with  $\lim_{n \rightarrow \infty} a_n = L$  then  $(a_n)_{n=1}^{\infty}$  is bounded. Is the converse true?

We can now justify our use of the word “diverges” in the situation that a sequence goes to  $\pm\infty$ .

**3.11.** Show that if  $(a_n)$  diverges to  $\infty$  then  $a_n$  diverges. Show that if  $(a_n)$  diverges to  $-\infty$ , it also diverges.

**3.12.** Let  $(b_n)_{n=1}^\infty$  be a convergent sequence with  $\lim_{n \rightarrow \infty} b_n = M$  with  $M \neq 0$ . Prove that there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$  we have  $|b_n| > \frac{|M|}{2}$ .

The following are often called the “Limit Laws” for sequences.

**3.13.** Let  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  be sequences and  $c \in \mathbb{R}$  be an arbitrary real number. We can define new sequences  $(c \cdot a_n)_{n=1}^\infty$ ,  $(a_n + b_n)_{n=1}^\infty$ , and  $(a_n \cdot b_n)_{n=1}^\infty$ . If  $b_n \neq 0$  for all  $n \in \mathbb{N}$  then we can define  $(\frac{a_n}{b_n})_{n=1}^\infty$ .

Prove that:

- (1) If  $(a_n)_{n=1}^\infty$  is a convergent sequence and  $c \in \mathbb{R}$  then  $(c \cdot a_n)_{n=1}^\infty$  is a convergent sequence and

$$\lim_{n \rightarrow \infty} (c \cdot a_n) = c \cdot \lim_{n \rightarrow \infty} a_n.$$

- (2) If  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are convergent sequences then  $(a_n + b_n)_{n=1}^\infty$  is a convergent sequence and

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n.$$

- (3) If  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are convergent sequences then  $(a_n \cdot b_n)_{n=1}^\infty$  is a convergent sequence and

$$\lim_{n \rightarrow \infty} (a_n \cdot b_n) = \left( \lim_{n \rightarrow \infty} a_n \right) \cdot \left( \lim_{n \rightarrow \infty} b_n \right).$$

- (4) If  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are convergent sequences with  $b_n \neq 0$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} b_n \neq 0$  then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}.$$

Hint for (3): The problem in (3) is that we have two quantities changing simultaneously. To deal with this we use a very common trick in analysis: we add and subtract additional terms, which does not affect the value, and then group terms so that each term is a product of things we can control. Let  $L = \lim_{n \rightarrow \infty} a_n$  and  $M = \lim_{n \rightarrow \infty} b_n$ . We can write

$$\begin{aligned} a_n \cdot b_n - L \cdot M &= a_n \cdot b_n - L \cdot M + L \cdot b_n - L \cdot b_n \\ &= (a_n - L) \cdot b_n + L \cdot (b_n - M). \end{aligned}$$

Hint for (4): Given (3) it suffices to prove (explain why) that

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{1}{\lim_{n \rightarrow \infty} b_n}.$$

Let  $M = \lim_{n \rightarrow \infty} b_n$  and notice that

$$\left| \frac{1}{b_n} - \frac{1}{M} \right| = \frac{|M - b_n|}{|Mb_n|} < \frac{|M - b_n|}{(M^2/2)}$$

if  $|b_n| > \frac{|M|}{2}$ . Now use Problem 3.12.

**3.14.** Suppose  $a \leq a_n \leq b$  for all  $n \in \mathbb{N}$ . Prove that if  $\lim_{n \rightarrow \infty} a_n = L$ , then  $L \in [a, b]$ .

**3.15.** Let  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  be convergent sequences with  $a = \lim_{n \rightarrow \infty} a_n$  and  $b = \lim_{n \rightarrow \infty} b_n$ . Prove that if  $a_n - b_n = \frac{1}{n}$  for all  $n \in \mathbb{N}$  then  $a = b$ . Would this theorem still be true if, instead of equality, you had  $a_n - b_n < \frac{1}{n}$ ? What if  $a_n - b_n = \frac{1}{2^n}$ ?

#### 4. Monotone Sequences; least upper bound and greatest lower bound

In general it can be difficult to show that a sequence converges. For certain classes of sequences checking for convergence can be much easier.

**Definition.** Let  $(a_n)$  be a sequence. We say

- (1)  $(a_n)$  is **increasing** if for all  $n \in \mathbb{N}$ ,  $a_n \leq a_{n+1}$ .
- (2)  $(a_n)$  is **decreasing** if for all  $n \in \mathbb{N}$ ,  $a_n \geq a_{n+1}$ .
- (3)  $(a_n)$  is **monotone** if it is increasing or decreasing.

Before proceeding we need some new definitions.

**Definition.** Let  $A \subseteq \mathbb{R}$  and  $x \in \mathbb{R}$ .

- (1)  $x$  is called an **upper bound for**  $A$  if, for all  $a \in A$ , we have  $a \leq x$ .
- (2)  $x$  is called an **lower bound for**  $A$  if, for all  $a \in A$ , we have  $x \leq a$ .
- (3) The set  $A$  is called **bounded above** if there exists an upper bound for  $A$ .  
The set  $A$  is called **bounded below** if there exists a lower bound for  $A$ . The set  $A$  is called **bounded** if it is both bounded above and bounded below.
- (4)  $x$  is called a **maximum of**  $A$  if  $x \in A$  and  $x$  is an upper bound for  $A$ .
- (5)  $x$  is called a **minimum of**  $A$  if  $x \in A$  and  $x$  is a lower bound for  $A$ .

In (4) ((5)) we write  $x = \max A$  ( $\min A$ ). Note there is only one maximum (or minimum), if any.

**3.16.** For each of the following subsets of  $\mathbb{R}$ :

- (a)  $A = \emptyset$ ,
- (b)  $A = \{a \in \mathbb{R} : 0 < a \leq 1\}$ ,
- (c)  $A = \{a \in \mathbb{R} : 2 \leq a\}$ .
- (d)  $A = \{a \in \mathbb{R} : a^2 < 2\}$ ,

- (1) Find all lower bounds for  $A$  and all upper bounds for  $A$ ,

- (2) Find  $\min A$  and  $\max A$  if they exist.
- (3) Discuss whether  $A$  is bounded.

**Definition.** Let  $A \subseteq \mathbb{R}$  and  $x \in \mathbb{R}$

- (1)  $x$  is called a **least upper bound of  $A$**  or **supremum of  $A$**  if
  - (a)  $x$  is an upper bound for  $A$ ,
  - (b) if  $y$  is an upper bound for  $A$  then  $x \leq y$ .
- (2)  $x$  is called a **greatest lower bound of  $A$**  or **infimum of  $A$**  if
  - (a)  $x$  is a lower bound for  $A$ ,
  - (b) if  $y$  is a lower bound for  $A$  then  $y \leq x$ .

**3.17.** Let  $A \subseteq \mathbb{R}$ . Prove that if  $x$  is a supremum of  $A$  and  $y$  is a supremum of  $A$  then  $x = y$ .

Hence, if the set  $A$  has a supremum then that supremum is unique and we can speak of *the* supremum of  $A$  and write  $\sup A$ . Similarly, if the set  $A$  has an infimum then that infimum is unique and we can speak of *the* infimum of  $A$  and write  $\inf A$ .

**3.18.** Find  $\inf A$  and  $\sup A$ , if they exist, for each of the following subsets of  $\mathbb{R}$ :

- (a)  $A = \emptyset$ ,
- (b)  $A = \{a \in \mathbb{R} : 0 < a \leq 1\}$ ,
- (c)  $A = \{a \in \mathbb{R} : 2 \leq a\}$ .
- (d)  $A = \{a \in \mathbb{R} : a^2 < 2\}$ ,

The above definitions could also be made in  $\mathbb{Q}$  or in any other subset of  $\mathbb{R}$ . Of course the answers to the exercises would depend upon the universe in question. For example if  $A$  is the set of all positive irrationals, then  $\inf A$  would not exist inside of the universe  $\mathbb{R} \setminus \mathbb{Q}$ .

**The Completeness Axiom:**  $\mathbb{R}$  is complete. That is, if  $A \subseteq \mathbb{R}$ ,  $A \neq \emptyset$  and  $A$  is bounded above, then  $\sup A$  exists.

Though  $\sup A$  need not be in the set  $A$  there are elements of  $A$  arbitrarily close to  $\sup A$ .

**3.19.** Let  $A \subseteq \mathbb{R}$  be non-empty and bounded above. Prove that if  $s = \sup A$  then for every  $\varepsilon > 0$  there exists an  $a \in A$  with  $s - \varepsilon < a \leq s$ .

What would the equivalent property of  $\inf A$  be?

**3.20.** Let  $A \subseteq \mathbb{R}$  be bounded above and non-empty. Let  $s = \sup A$ . Prove that there exists a sequence  $(a_n) \subseteq A$  with  $\lim_n a_n = s$ . Prove that, in addition, the sequence  $(a_n)$  can be chosen to be increasing, i.e.,  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$ .

What would be the analogous results for  $t = \inf A$ , if  $A$  is bounded below?

**3.21.** Let  $A \subseteq \mathbb{R}$ . Define

$$-A = \{-x : x \in A\}.$$

- (1) Prove that  $x$  is an upper bound for  $A$  if and only if  $-x$  is a lower bound for  $-A$ .
- (2) Prove that  $x = \sup A$  if and only if  $-x = \inf -A$ .
- (3) Prove that if  $A$  is non-empty and bounded below then  $\inf A$  exists.

**3.22.** Prove or disprove:

- (1) If  $A, B \subseteq \mathbb{R}$  are nonempty sets such that for every  $a \in A$  and for every  $b \in B$  we have  $a \leq b$  then  $\sup A$  and  $\inf B$  exist and  $\sup A \leq \inf B$ .
- (2) If  $A, B \subseteq \mathbb{R}$  are nonempty sets such that for every  $a \in A$  and for every  $b \in B$  we have  $a < b$ , then  $\sup A$  and  $\inf B$  exist and  $\sup A < \inf B$ .

**3.23.** Let  $A, B \subset \mathbb{R}$  be non-empty, bounded sets. We define  $A + B = \{a + b : a \in A \text{ and } b \in B\}$ . Prove that

$$\begin{aligned} \sup(A + B) &= \sup(A) + \sup(B) \quad \text{and} \\ \inf(A + B) &= \inf(A) + \inf(B) \end{aligned}$$

**3.24.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $A \subseteq \mathbb{R}$ . Assume that  $f(A)$  and  $g(A)$  are bounded. Prove that

$$\sup(f + g)(A) \leq \sup f(A) + \sup g(A) .$$

Give examples where one has “=” and also “<”. State the analogous results for “inf”.

**3.25.** Prove the following:

- (1) If  $(a_n)$  is increasing and unbounded then  $(a_n)$  diverges to  $+\infty$ .
- (2) If  $(a_n)$  is decreasing and unbounded then  $(a_n)$  diverges to  $-\infty$ .
- (3) Let  $(a_n)$  be increasing and bounded. Let  $L = \sup\{a_n : n \in \mathbb{N}\}$ . Then  $\lim_{n \rightarrow \infty} a_n = L$ .
- (4) Let  $(a_n)$  be decreasing and bounded. Let  $L = \inf\{a_n : n \in \mathbb{N}\}$ . Then  $\lim_{n \rightarrow \infty} a_n = L$ .

**3.26.** Let  $A \subset \mathbb{R}$  be a non-empty, bounded set. Let  $\alpha = \sup(A)$  and  $\beta = \inf(A)$ , and let  $(a_n)_{n=1}^{\infty} \subset A$  be a convergent sequence, with  $a = \lim_{n \rightarrow \infty} a_n$ . Prove that  $\beta \leq a \leq \alpha$ . Notice that  $a$  need not be equal to either of  $\alpha$  or  $\beta$ , even if  $(a_n)$  is strictly increasing or strictly decreasing. For example, let  $A = [0, 1]$  so  $\alpha = 1$  and  $\beta = 0$ , and let  $a_n = \frac{1}{2} + \frac{1}{2^n}$ . Then  $(a_n)$  is strictly decreasing and  $a = \lim_{n \rightarrow \infty} a_n = \frac{1}{2}$ . If, instead,  $a_n = \frac{1}{2} - \frac{1}{2^n}$ , then  $(a_n)$  would be strictly increasing and  $a = \lim_{n \rightarrow \infty} a_n = \frac{1}{2}$ . In both cases,  $a \neq \alpha$  and  $a \neq \beta$ . Come up with a different example where  $(a_n)$  is monotonic (i.e., increasing or decreasing) but its limit is not  $\alpha$  or  $\beta$ .

**3.27.** If  $A \subset \mathbb{R}$  is a non-empty, bounded set and  $B \subset A$ , prove

$$\inf(A) \leq \inf(B) \leq \sup(B) \leq \sup(A) .$$

**3.28.** If  $A, B \subset \mathbb{R}$  are both non-empty, bounded sets, prove

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\} .$$

**3.29.** We analyze the important sequence  $(r^n)_{n=1}^\infty$ , where  $r \in \mathbb{R}$ .

(1) Prove that if  $0 \leq r < 1$  then  $(r^n)_{n=1}^\infty$  converges to 0.

**Hint:** Show the sequence is decreasing. Proceed by contradiction; suppose  $L = \inf\{a_n : n \in \mathbb{N}\} > 0$ , find  $a_n$  with  $L \leq a_n < r^{-1}L$ , and hence produce a contradiction.

(2) Prove that if  $r > 1$  then  $(r^n)_{n=1}^\infty$  diverges to  $+\infty$ .

(3) Use Problem 3.9 to complete the proof of the following fact: The sequence  $(r^n)$  converges if and only if  $-1 < r \leq 1$ . If  $r = 1$  then  $\lim_{n \rightarrow \infty} r^n = 1$ . If  $-1 < r < 1$  then  $\lim_{n \rightarrow \infty} r^n = 0$ .

The following example illustrates how powerful monotonicity is.

**3.30.** Let  $a_1 = 2$  and let  $(a_n)$  be generated by the recursive formula

$$a_{n+1} = \frac{1}{2} \left( a_n + \frac{2}{a_n} \right)$$

for  $n \geq 1$ .

(1) Prove that  $\sqrt{2} < a_n$  for all  $n \in \mathbb{N}$ .

**Hint:** Assuming  $a_n > 0$  turn  $\frac{1}{2} \left( a_n + \frac{2}{a_n} \right) > \sqrt{2}$  into an equivalent condition on a quadratic polynomial. Proceed by induction.

(2) Prove that  $(a_n)$  is decreasing and hence converges.

(3) Now take limits on both sides of the recursive formula using Problem 3.13 and use the fact that

$$\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n$$

to find  $\lim_{n \rightarrow \infty} a_n$ .

Let  $x \in \mathbb{R}^+$ . What can you say about the sequence given by  $a_1 = 1$  and  $a_{n+1} = \frac{1}{2} \left( a_n + \frac{x}{a_n} \right)$  for  $n \geq 1$ ?

## 5. Subsequences

If we have a sequence  $(a_n)$ , a subsequence is a (new) sequence formed by skipping (possibly infinitely many) terms in  $a_n$ .

**Definition.** A sequence  $(b_k)_{k=1}^\infty$  is a **subsequence** of  $(a_n)_{n=1}^\infty$  if there exists a strictly increasing sequence of natural numbers  $n_1 < n_2 < \dots$  so that for all  $k \in \mathbb{N}$ ,  $b_k = a_{n_k}$ .

**Example.**  $(1, 1, 1, \dots)$  is a subsequence of  $(1, -1, 1, -1, \dots)$ . In fact any sequence of  $\pm 1$ 's is a subsequence of  $(1, -1, 1, -1, \dots)$ .

In particular we see that a subsequence of a divergent sequence may be convergent.

**Example.**  $(\frac{1}{n^2})_{n=1}^{\infty}$  is a subsequence of  $(\frac{1}{n})_{n=1}^{\infty}$ .

Both  $(\frac{1}{n})_{n=1}^{\infty}$  and  $(\frac{1}{n^2})_{n=1}^{\infty}$  converge to 0.

**3.31.** Prove or disprove: If  $(b_n)$  is a subsequence of  $(a_n)$  and  $\lim_{n \rightarrow \infty} a_n = L$ , then  $\lim_{n \rightarrow \infty} b_n = L$ .

Now we can play a fun game to see whether we can find subsequences with better properties than the original sequence.

**3.32.** Prove that every sequence of real numbers has a monotone subsequence.

**Hint:** Consider two cases. The first is the case in which every subsequence has a minimum element. In this case we can extract an increasing subsequence.

**3.33.** Let  $(x_n)_{n=1}^{\infty}$  be an increasing sequence. Suppose that there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  that converges to a point  $x \in \mathbb{R}$ . Prove that  $(x_n)$  also converges to  $x$ .

Combining this with our earlier work on monotone sequences, Problem 3.25, we get a very useful result.

**3.34.** Prove that every bounded sequence of real numbers has a convergent subsequence.

**3.35.** Let  $(a_n)$  and  $(b_n)$  be sequences such that  $a_{n-1} \leq a_n < b_n \leq b_{n-1}$  for all  $n \in \mathbb{N}$ , and  $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$ . If we define  $I_n = [a_n, b_n]$  then  $I_1 \supset I_2 \supset I_3 \supset \dots$ .

1. Prove that there exists a  $p \in \mathbb{R}$  such that  $p \in I_n$  for all  $n \in \mathbb{N}$ .
2. Prove that if  $q \in I_n$  for all  $n \in \mathbb{N}$  then  $q = p$ .

## 6. Cauchy Sequences

The problem with the definition of a convergent sequence is that it requires us to know or guess what the limit is in order to prove a sequence converges. We saw when looking at bounded monotone sequences that sometimes it is possible to show that a sequence converges without knowing the limit.

We now give another definition that will address the problem.

**Definition.** A sequence  $(a_n)$  is called a **Cauchy** sequence if for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$  we have  $|a_n - a_m| < \varepsilon$ .

**3.36.** Negate the definition of Cauchy sequence to give an explicit definition of “ $(a_n)_{n=1}^{\infty}$  is not a Cauchy sequence.”

This will turn out to be very useful for proving that a sequence diverges.

**3.37.** Prove that every convergent sequence is Cauchy.

This is true much more generally than just for  $\mathbb{R}$ . It is particularly useful in the contrapositive form; if  $(a_n)_{n=1}^{\infty}$  is not Cauchy then  $(a_n)_{n=1}^{\infty}$  diverges.

**3.38.** Prove that every Cauchy sequence is bounded. Is the converse true?

**3.39.** Let  $(a_n)$  be a Cauchy sequence and let  $(a_{n_k})$  be a convergent subsequence. Prove that  $(a_n)$  is convergent and  $\lim_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} a_{n_k}$ .

Again this statement does not depend on special properties of  $\mathbb{R}$ . However, Problem 3.39 together with what we know about sequences in  $\mathbb{R}$  proves the converse of Problem 3.37 is true for sequences in  $\mathbb{R}$ .

**3.40.** Let  $(a_n)$  be Cauchy sequence in  $\mathbb{R}$ . Prove that  $(a_n)$  is convergent.

A Cauchy sequence in  $\mathbb{R}$  is thus the same as a convergent sequence in  $\mathbb{R}$ . The advantage is that the definition of a Cauchy sequence makes no reference to the limit; it is an intrinsic property of the sequence.

This fact does not hold in every “universe.” Is every Cauchy sequence in  $\mathbb{Q}$  convergent to some point in  $\mathbb{Q}$ ?

**3.41.** Prove or give a counterexample: If a sequence of real numbers  $(x_n)_{n=1}^{\infty}$  has the property that for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$  we have  $|x_{n+1} - x_n| < \varepsilon$ , then  $(x_n)$  is a convergent sequence. How is this different from the definition of a Cauchy sequence?

## 7. Decimals

**3.42.** The way that many people think of real numbers is as decimal expansions. Of course the only decimal expansions that we can easily write down either terminate

$$\frac{1}{8} = 0.125 \quad \frac{27}{50} = 0.54$$

or become periodic

$$\frac{1}{3} = 0.3333 \dots = 0.\overline{3} \quad \frac{1}{7} = 0.142857142857 \dots = 0.\overline{142857}$$

Write down a decimal that is not periodic in such a way that the pattern is clear. It is an interesting fact that a decimal expansion which either terminates or becomes periodic represents a rational number. Unfortunately some numbers have two decimal expansions. For example we could write

$$\frac{1}{8} = 0.124\overline{9}.$$

All the numbers that have multiple decimal expansions are rational. However, not all rational numbers have multiple expansions.

Shortly we will prove that  $\pi$  has a decimal expansion even though not all the digits are known; as of this writing the first 1,241,100,000,000 decimal digits are known. They were calculated by the laboratory of Yasumasa Kanada at the University of Tokyo. Chao Lu of China holds the Guinness Book of Records record for reciting digits of  $\pi$ ; he recited 67,890 digits.

Suppose  $x \in [0, 1]$ .

For a finite sequence  $n_1, \dots, n_k$  with  $n_i \in \{0, 1, \dots, 9\}$  we can define a closed subinterval of  $[0, 1]$  by

$$I_{n_1, \dots, n_k} = \left[ \frac{n_1}{10} + \frac{n_2}{100} + \dots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k}, \frac{n_1}{10} + \frac{n_2}{100} + \dots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k + 1}{10^k} \right]$$

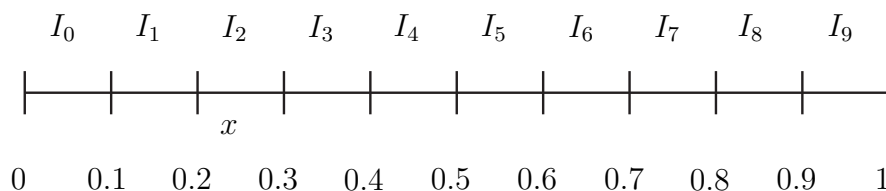


FIGURE 1.  $[0, 1]$  is the union of the intervals  $I_0, \dots, I_9$ . In this case  $x \in I_2$ .

First we notice that

$$[0, 1] = \bigcup_{i=0}^9 I_i$$

and hence there exists  $n_1 \in \{0, 1, \dots, 9\}$  such that  $x \in I_{n_1}$ .

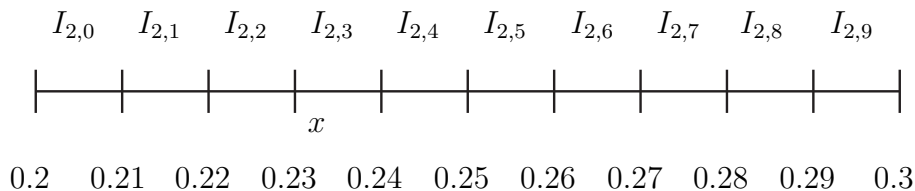


FIGURE 2. Since  $x \in I_2$  we subdivide  $I_2$ .

Now we suppose that we have  $n_1, \dots, n_k$  with  $n_i \in \{0, 1, \dots, 9\}$  such that  $x \in I_{n_1, \dots, n_k}$ . Notice that

$$I_{n_1, \dots, n_k} = \bigcup_{i=0}^9 I_{n_1, \dots, n_k, i}$$

and hence there exists  $n_{k+1} \in \{0, 1, \dots, 9\}$  such that  $x \in I_{n_1, \dots, n_k, n_{k+1}}$ .

This process defines an infinite sequence  $n_1, n_2, n_3, \dots$  and the decimal representation of  $x$  is then  $.n_1 n_2 n_3 \dots$ . In general, if  $x \in \mathbb{R}$ , we let  $n_0$  be the largest integer less than or equal to  $x$ . Then the decimal expansion for  $x$  is  $n_0.n_1 n_2 n_3 \dots$ , where  $0.n_1 n_2 n_3 \dots$  is the decimal expansion for  $x - n_0 \in [0, 1]$ .

**3.43.**

- (1) For each
- $k \in \mathbb{N}$
- define
- $x_k \in \mathbb{Q}$
- by

$$x_k = n_0.n_1 \dots n_k.$$

Prove that  $\lim_{k \rightarrow \infty} x_k = x$ .

- (2) Explain how a real number  $x$  may have two decimal expansions.
- (3) Suppose  $x, y \in \mathbb{R}$  and suppose  $(a_k)$  and  $(b_k)$  are decimal expansions for  $x$  and  $y$  respectively. In addition, assume neither  $(a_k)$  nor  $(b_k)$  ends with a constant sequence of 9's (see the previous question). Show that  $x < y$  if and only if there exists a  $k \in \mathbb{N}$  with  $a_0.a_1a_2 \dots a_k < b_0.b_1b_2 \dots b_k$ .
- (4) Suppose  $x, y \in \mathbb{R}$  and suppose  $\{a_k\}$  and  $\{b_k\}$  are decimal expansions for  $x$  and  $y$  respectively. Describe (and prove) how to find a decimal expansion for  $x + y$ .
- (5) Prove that a decimal expansion is eventually periodic if and only if it comes from a rational number.

**3.44.** Let  $(d_n)_{n=1}^{\infty}$  be an arbitrary sequence with  $d_n \in \{0, 1, \dots, 9\}$  for each  $n \in \mathbb{N}$ . Define a sequence  $(a_n)_{n=1}^{\infty}$  by

$$a_n = 0.d_1d_2 \dots d_n = \sum_{k=1}^n \frac{d_k}{10^k}.$$

Prove that  $(a_n)_{n=1}^{\infty}$  converges to a real number. Hence, give another proof that for any  $a, b \in \mathbb{R}$  with  $a < b$  there exists a rational number  $x \in (a, b)$ .



## CHAPTER 4

# Limits and Continuity

### 1. Limits

The intuitive idea of a number  $L \in \mathbb{R}$  being the limit of a function  $f(x)$  as  $x$  approaches a point  $p$  is that, for all  $x$  close enough to  $p$ , the value of the function is as close as we like to  $L$ . The limit should not depend on the value of  $f$  at  $p$  but only on the value of  $f$  at points  $x$  near  $p$ . Indeed for a limit to exist at  $p$  it is not even necessary that  $f$  be defined at  $p$  but only that  $f$  be defined at points  $x$  near  $p$ .

**Definition.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  a function,  $p \in \bar{I}$ , and  $L \in \mathbb{R}$ . We say that  $L$  is a **limit of  $f$  as  $x$  approaches  $p$**  if, for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in I$  if  $0 < |x - p| < \delta$  then  $|f(x) - L| < \varepsilon$ .

When proving that  $L$  is a limit of  $f$  as  $x$  approaches  $p$ , we are given an arbitrary  $\varepsilon > 0$  and have to find a  $\delta > 0$  exactly as we had to find a  $N \in \mathbb{N}$  when proving that  $L$  was the limit of a sequence. In practice this means that we seek to estimate  $|f(x) - L|$  from above making it  $< \varepsilon$  using  $|x - p| < \delta$ .

This definition gives you no way of finding a limit  $L$ .

Exactly as for sequences, Problem 3.2, we have to show that limits are in fact unique.

**4.1.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  a function, and  $p \in \bar{I}$ . Suppose  $L$  and  $M$  are both limits of  $f$  as  $x$  approaches  $p$ . Show that  $L = M$ .

This shows that if a limit of  $f$  as  $x$  approaches  $p$  exists then it is unique. Now we can talk about *the* limit of  $f$  as  $x$  approaches  $p$  and write  $\lim_{x \rightarrow p} f(x) = L$ .

**Note:** When we write  $\lim_{x \rightarrow p} f(x) = L$  we are making two assertions; the limit of  $f$  as  $x$  approaches  $p$  exists, and its value is  $L$ . Exactly as with sequences we must take care never to write  $\lim_{x \rightarrow p} f(x)$  until after we have shown that the limit exists.

If the point  $p$  can be approached from both sides by points of  $I$ , that is if  $p \in I^\circ$ , then we can define the **left-hand limit** and **right-hand limit** of  $f$  as  $x$  approaches  $p$ .

**Definition.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  a function,  $p \in \bar{I}$ , and  $L \in \mathbb{R}$ . We say that  $L$  is a **right-hand limit of  $f$  as  $x$  approaches  $p$**  if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $I \cap (p, p + \delta) \neq \emptyset$ , and for all  $x \in I$  if  $p < x < p + \delta$  then  $|f(x) - L| < \varepsilon$ . We say that  $L$  is a **left-hand limit of  $f$  as  $x$  approaches  $p$**  if for

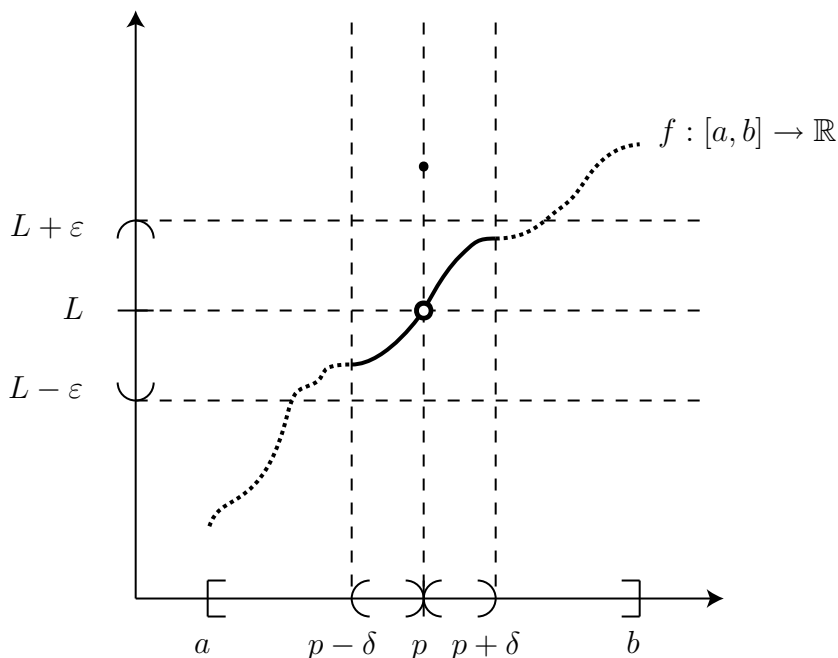


FIGURE 1. Showing that the limit of  $f : [a, b] \rightarrow \mathbb{R}$  as  $x$  approaches  $p$  is  $L$ ; a possible choice of  $\delta > 0$  for the given  $\varepsilon > 0$ .

every  $\varepsilon > 0$  there exists a  $\delta > 0$ , such that  $I \cap (p - \delta, p) \neq \emptyset$ , and for all  $x \in I$  if  $p - \delta < x < p$  then  $|f(x) - L| < \varepsilon$ .

Similarly left-hand and right-hand limits are unique (why) and are denoted by  $\lim_{x \rightarrow p^-} f(x) = L$  and  $\lim_{x \rightarrow p^+} f(x) = L$  respectively.

**4.2.** Let  $I \subseteq \mathbb{R}$ ,  $f : I \rightarrow \mathbb{R}$  a function, and  $p \in I^\circ$ . Prove that  $\lim_{x \rightarrow p} f(x) = L$  if and only if  $\lim_{x \rightarrow p^-} f(x) = \lim_{x \rightarrow p^+} f(x) = L$ .

**4.3.** Let  $I \subseteq \mathbb{R}$  be an interval and  $p \in \bar{I}$ . Give the negation of the definition of “ $\lim_{x \rightarrow p} f(x) = L$ ”.

**Note:** The negation of “ $\lim_{x \rightarrow p} f(x) = L$ ” is not “ $\lim_{x \rightarrow p} f(x) \neq L$ ” since that implies the existence of the limit. In words we would phrase the negation as  $f(x)$  does not approach  $L$  as  $x$  approaches  $p$ . There are two possibilities;  $\lim_{x \rightarrow p} f(x)$  exists but  $\lim_{x \rightarrow p} f(x) \neq L$ , or  $f$  has no limit as  $x$  approaches  $p$ .

**4.4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = 3$  for all  $x \in \mathbb{R}$ . Let  $p \in \mathbb{R}$  be arbitrary. Prove that  $\lim_{x \rightarrow p} f(x) = 3$ .

**Remark.** In this case the  $\delta$  that you get does not depend on  $\varepsilon$  for any  $p$ . For a function  $f$  defined on an interval this only occurs when  $f$  is constant.

**4.5.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = x$  for all  $x \in \mathbb{R}$ . Let  $p \in \mathbb{R}$ . Prove  $\lim_{x \rightarrow p} f(x) = p$ .

**Remark.** Now we see that  $\delta$  depends on  $\varepsilon$  and that  $\delta$  goes to 0 as  $\varepsilon$  goes to 0. However the choice of  $\delta$  still does not depend on the choice of  $p$ .

**4.6.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = 3x - 5$  for all  $x \in \mathbb{R}$ . Let  $p \in \mathbb{R}$ . Prove  $\lim_{x \rightarrow p} f(x) = 3p - 5$ .

**4.7.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = x^2$  for all  $x \in \mathbb{R}$ . Let  $p \in \mathbb{R}$ . Prove  $\lim_{x \rightarrow p} f(x) = p^2$ .

**Hint:** Start the proof by writing down what needs to be shown. The goal is always to estimate  $|f(x) - p^2|$  from above by some function of  $\delta$  that goes to 0 as  $\delta$  goes to 0. By the difference of squares formula,  $|x^2 - p^2| = |x + p| |x - p|$ . The definition of the limit gives us the estimate  $|x - p| < \delta$ . However, since our choice of  $\delta$  is only allowed to depend on  $\varepsilon$  and  $p$  but not on  $x$  we must estimate  $|x + p|$  by some quantity independent of  $x$ .

Since we know that  $|x - p| < \delta$  we can say  $|x| < |p| + \delta$  and hence we can estimate

$$|x + p| \leq |x| + |p| < 2|p| + \delta$$

and hence

$$|x^2 - p^2| < (2|p| + \delta)\delta.$$

Now if we fix an  $\varepsilon > 0$  then we can always choose  $\delta > 0$  such that  $(2|p| + \delta)\delta < \varepsilon$ . If we complete the square or use the quadratic formula then we find an “optimal” choice for  $\delta$ . However, it is important to remember that we don’t need to find the best  $\delta$  we just need to find a  $\delta > 0$  that works. If we knew  $\delta \leq 1$  then we could estimate  $|x + p| < 2|p| + 1$  and hence  $|x^2 - p^2| < (2|p| + 1)\delta$ . From this it is easy to choose a  $\delta > 0$  such that both  $\delta \leq 1$  and  $(2|p| + 1)\delta < \varepsilon$ .

**Note:** Regardless of how we produce our choice of  $\delta$  it depends on the point  $p$ . For a fixed  $\varepsilon$  we see that the  $\delta$  required gets smaller and smaller as  $|p|$  gets bigger and bigger. How could you tell this from looking at the graph of  $f$ ?

**4.8.** Using the definition of the limit of a function (i.e., the  $\varepsilon$ - $\delta$  definition), prove that  $\lim_{x \rightarrow 2} (2x^2 - x + 1) = 7$ .

**4.9.** Let  $I \subset \mathbb{R}$  be an interval and let  $f : I \rightarrow \mathbb{R}$ , with  $p \in \bar{I}$ . Assume there exist  $a < b$  and  $\delta > 0$  such that for all  $x \in I$ , if  $|x - p| < \delta$  then  $f(x) \in [a, b]$ . Prove that if  $L = \lim_{x \rightarrow p} f(x)$  exists, then  $L \in [a, b]$ .

**4.10.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational.} \end{cases}$$

Prove that  $\lim_{x \rightarrow p} f(x)$  does not exist for any  $p \in \mathbb{R}$ .

**4.11.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ x & x \text{ is rational.} \end{cases}$$

Prove that  $\lim_{x \rightarrow p} f(x)$  exists for only one value of  $p$ .

**4.12.** Let  $f : [a, b] \rightarrow \mathbb{R}$  and let  $p \in [a, b]$ . Assume  $\lim_{x \rightarrow p} f(x) = L$  exists with  $L > 0$ . Prove that there exists  $\delta > 0$  such that if  $x \in [a, b]$  and  $0 < |x - p| < \delta$  then  $f(x) > L/2$ .

The next problem relates the limit of a function as  $x$  approaches  $p$  to the limits of sequences.

**4.13.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  a function,  $p \in \bar{I}$ , and  $L \in \mathbb{R}$ . Prove that  $\lim_{x \rightarrow p} f(x) = L$  if and only if for every sequence  $(x_n) \subseteq I \setminus \{p\}$  with  $\lim_{n \rightarrow \infty} x_n = p$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = L$ .

**Hint:** To prove “if for every sequence  $(x_n) \subset I \setminus \{a\}$  with  $\lim_{n \rightarrow \infty} x_n = p$  we have  $\lim_{n \rightarrow \infty} f(x_n) = L$ ” you should switch to the contrapositive.

**4.14.** State the contrapositive of the sequential characterization, Problem 4.13, of limits (i.e., get a new if and only if statement by negating both sides).

This statement is quite useful for proving that a function has no limit as  $x$  approaches  $p$ .

**4.15.** Which of these limits (if any) exist? Prove your answer.

- (1)  $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ .
- (2)  $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$ .

The following are the analogues of the limit laws for sequences, Problem 3.13.

**4.16.** Let  $I \subseteq \mathbb{R}$  be an interval,  $p \in \bar{I}$ , and  $f : I \rightarrow \mathbb{R}$  and  $g : I \rightarrow \mathbb{R}$  functions satisfying

$$\lim_{x \rightarrow p} f(x) = L \quad \lim_{x \rightarrow p} g(x) = M$$

Let  $c \in \mathbb{R}$ . Prove that

- (1)  $\lim_{x \rightarrow p} c \cdot f(x) = c \cdot L$ .

- (2)  $\lim_{x \rightarrow p} (f(x) + g(x)) = L + M$ .  
 (3)  $\lim_{x \rightarrow p} f(x) \cdot g(x) = L \cdot M$ .  
 (4) If  $g(x) \neq 0$  for  $x \in I$  and  $M \neq 0$  then  $\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \frac{L}{M}$ .

There are two ways to prove these statements; one is to use the definition of limit directly, and the other is to use the sequential characterization of the limit, Problem 4.13, and our earlier limit theorems for sequences, Problem 3.13.

**Remark.** The condition  $g(x) \neq 0$  for  $x \in I$  is stronger than necessary. It ensures that the function  $\frac{f(x)}{g(x)}$  is defined for all  $x \in I$ . If  $\lim_{x \rightarrow p} g(x) \neq 0$  then there exists an interval  $J \subseteq I$  with  $p \in \bar{J}$  and  $g(x) \neq 0$  for all  $x \in J$ .

**4.17.** Let  $I \subset \mathbb{R}$  be an interval, and let  $p \in I$ . Let  $f$ ,  $g$ , and  $h$  be functions on  $I \setminus \{p\}$  such that  $g(x) \leq f(x) \leq h(x)$  for all  $x \in I \setminus \{p\}$ . Prove that if  $\lim_{x \rightarrow p} g(x) = \lim_{x \rightarrow p} h(x) = L \in \mathbb{R}$ , then  $\lim_{x \rightarrow p} f(x) = L$ . (This is the Squeeze Theorem for functions.)

## 2. Continuous Functions

**Definition.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  a function, and  $p \in I$ . We say that  **$f$  is continuous at  $p$**  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that if  $x \in I$  and  $|x - p| < \delta$  then  $|f(x) - f(p)| < \varepsilon$ .

If  $f$  is continuous at all  $p \in I$  it is called **continuous**. If  $S \subseteq I$  and  $f$  is continuous at all  $p \in S$  it is called **continuous on  $S$** .

**4.18.** Let  $f : I \rightarrow \mathbb{R}$  where  $I$  is an interval and let  $p \in I$ . Negate the definition of “ $f$  is continuous at  $p$ ”.

**4.19.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find all points  $p \in \mathbb{R}$  at which  $f$  is continuous. Justify your answer.

**4.20.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational.} \end{cases}$$

Find all points  $p \in \mathbb{R}$  at which  $f$  is continuous. Justify your answer.

**4.21.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ x & x \text{ is rational.} \end{cases}$$

Find all points  $p \in \mathbb{R}$  at which  $f$  is continuous. Justify your answer.

The definition of  $f$  being continuous at  $p$  is very similar to the definition of the limit of  $f$  as  $x$  approaches  $p$ . The following theorem makes the connection explicit. This is probably the definition of continuity that you saw in your Calculus class.

**4.22.** Let  $I \subset \mathbb{R}$  be an interval and  $p \in I$ . Prove that  $f$  is continuous at  $p$  if and only if  $\lim_{x \rightarrow p} f(x) = f(p)$ .

Since we have a sequential characterization of the limit of  $f$  as  $x$  approaches  $p$ , Problem 4.13, we can obviously produce a sequential characterization of continuity.

**4.23.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  a function, and  $p \in I$ . Prove that  $f$  is continuous at  $p$  if and only if for every sequence  $(x_n) \subseteq I$  with  $\lim_{n \rightarrow \infty} x_n = p$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = f(p)$ .

This is slightly different from a direct application of Problem 4.13 since we have every sequence  $(x_n) \subseteq I$  rather than every sequence  $(x_n) \subseteq I \setminus \{p\}$ . Is the theorem still true if we replace “every sequence  $(x_n) \subseteq I$ ” with every sequence  $(x_n) \subseteq I \setminus \{p\}$ ?

**4.24.** Give the contrapositive to the sequential characterization of continuity, Problem 4.23.

The limit laws for functions, Problem 4.16, can be reinterpreted in terms of continuous functions.

**4.25.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  and  $g : I \rightarrow \mathbb{R}$  functions, and  $p \in I$ . Assume that  $f$  and  $g$  are continuous at  $p$ . Let  $c \in \mathbb{R}$ . Prove

- (1)  $f + g$  is continuous at  $p$ .
- (2)  $c \cdot f$  is continuous at  $p$ .
- (3)  $f \cdot g$  is continuous at  $p$ .
- (4) If  $g(x) \neq 0$  for  $x \in I$  then  $\frac{f(x)}{g(x)}$  is continuous at  $p$ .

**4.26.** Prove that every polynomial function is continuous on  $\mathbb{R}$ .

There is one more operation which we can perform with functions that has no direct analogue for sequences.

**4.27.** Let  $I, J \subseteq \mathbb{R}$  be intervals,  $f : I \rightarrow \mathbb{R}$  and  $g : J \rightarrow \mathbb{R}$  functions with  $f(I) \subseteq J$ . Let  $p \in I$ . Prove that if  $f$  is continuous at  $p$  and  $g$  is continuous at  $f(p)$  then  $g \circ f$  is continuous at  $p$ .

This can be proved either directly from the definition or by repeated application of Problem 4.23.

One of the most important theorems about continuous functions on intervals is the “Intermediate Value Theorem”.

**4.28.** If  $f$  is a continuous function on  $[a, b]$  with  $a < b$  and  $f(a) < y < f(b)$  or  $f(a) > y > f(b)$  then there exists  $p \in (a, b)$  with  $f(p) = y$ .

**Hint:** Suppose  $f(a) < y < f(b)$  and let  $E = \{x \in [a, b] : f(x) < y\}$ . Let  $p = \sup E$ . The point  $p$  can be written as the limit of a sequence  $x_n \in E$  and as the limit of a sequence  $x'_n \in [a, b] \setminus E$ . Hence prove that  $f(p) = y$ .

**4.29.** Let  $I \subseteq \mathbb{R}$  be any interval and  $f : I \rightarrow \mathbb{R}$  a non-constant continuous function. Prove that  $f(I)$  is an interval.

**Remark.** First prove that it suffices to show that given any two points  $c, d \in f(I)$  the entire interval between them is contained in  $f(I)$ .

In general we cannot say any more about the interval  $f(I)$ .

**Definition.** We say that a function  $f : I \rightarrow \mathbb{R}$  is **bounded** if the set  $f(I)$  is bounded. Thus  $f$  is bounded if and only if there exists  $K \in \mathbb{R}$  such that  $|f(x)| \leq K$  for all  $x \in I$ .

**4.30.** Give an example of a continuous function  $f : (0, 1) \rightarrow \mathbb{R}$  that is not bounded.

Thus the continuous image of a bounded interval may be unbounded.

**4.31.** Give an example of a continuous function  $f : (0, 1) \rightarrow \mathbb{R}$  such that  $f((0, 1))$  is a closed and bounded interval.

Thus the continuous image of an open interval may be a closed interval.

However, in the special case of a continuous function on a closed and bounded interval we can say a lot more.

**4.32.** Let  $f$  be a continuous function on  $[a, b]$  with  $a < b$ . Show that  $f$  is bounded.

**Hint:** Proceed by contradiction. Suppose that  $f$  is not bounded above and construct a sequence  $(x_n)_{n=1}^{\infty}$  with  $x_n \in [a, b]$  for all  $n \in \mathbb{N}$  such that

$$\lim_{n \rightarrow \infty} f(x_n) = +\infty$$

Now apply the sequential characterization of continuity, Problem 4.23, to obtain a contradiction.

**4.33.** Let  $f$  be a continuous function on  $[a, b]$  with  $a < b$ . Show that there exist  $x_m, x_M \in [a, b]$  such that  $f(x_m) \leq f(x) \leq f(x_M)$  for all  $x \in [a, b]$ . We say  $f$  achieves its **maximum** and **minimum** value.

**Hint:** Construct a sequence  $(x_n)_{n=1}^{\infty}$  with  $x_n \in [a, b]$  for all  $n \in \mathbb{N}$  such that

$$\lim_{n \rightarrow \infty} f(x_n) = \sup\{f(x) : x \in [a, b]\}.$$

Now apply the sequential characterization of continuity, Problem 4.23, to find a point  $p \in [a, b]$  with  $f(p) = \sup\{f(x) : x \in [a, b]\}$ .

**4.34.** Give an example of  $f : (0, 1) \rightarrow \mathbb{R}$  that is bounded, continuous, and has neither a maximum nor a minimum. Can you do the same for  $f : (0, 1] \rightarrow \mathbb{R}$ ?

### 3. Uniform Continuity

Sometimes we encounter a property stronger than continuity. Recall that  $f : I \rightarrow \mathbb{R}$  is continuous on  $I$ , if for all  $p \in I$  and for all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in I$  if  $|x - p| < \delta$  then  $|f(x) - f(p)| < \varepsilon$ .

For a continuous function, the  $\delta$  generally depends upon both  $\varepsilon$  and the point  $p$  as previous exercises have illustrated. If we remove the dependence on  $p$  we have uniform continuity.

**Definition.** Let  $I \subseteq \mathbb{R}$  be an interval and  $f : I \rightarrow \mathbb{R}$  a function. We say that  $f$  is **uniformly continuous** on  $I$  if for all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in I$  if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \varepsilon$ .

**4.35.** Suppose  $I \subseteq \mathbb{R}$  is an interval. Prove that if  $f : I \rightarrow \mathbb{R}$  is uniformly continuous on  $I$  then  $f$  is continuous on  $I$ .

**4.36.** Let  $f(x) = mx + c$  for some  $m, c \in \mathbb{R}$ . Prove that  $f(x)$  is uniformly continuous on  $\mathbb{R}$ .

**4.37.** Negate the definition of uniform continuity.

**4.38.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = x^2$ . Prove that  $f$  is not uniformly continuous on  $\mathbb{R}$ .

**Hint:** Fix an  $\varepsilon > 0$  and show that no matter what  $\delta > 0$  is chosen you can always choose  $x, y \in \mathbb{R}$  such that  $|x - y| < \delta$  and  $|x^2 - y^2| \geq \varepsilon$ .

**4.39.** Prove that if  $I$  is a bounded interval and  $f : I \rightarrow \mathbb{R}$  is uniformly continuous then  $f$  is bounded.

This together with Problem 4.30 shows that we can have continuous functions on  $(0, 1)$  that are not uniformly continuous. As in the previous section the case of functions on closed and bounded intervals is very different.

**4.40.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Prove that  $f$  is uniformly continuous.

**Hint:** Suppose that  $f$  is not uniformly continuous. Then there exists an  $\varepsilon > 0$  such that for every  $\delta > 0$  there exists  $x, y \in [a, b]$  such that  $|x - y| < \delta$  but  $|f(x) - f(y)| \geq \varepsilon$ . Show that there exists two sequences  $(x_n), (y_n) \subset [a, b]$  which both converge to the same point  $p \in [a, b]$  but such that  $|f(x_n) - f(y_n)| \geq \varepsilon$ . Hence conclude that  $f$  is not continuous at  $p$ .

## CHAPTER 5

### Differentiation

#### 1. Derivatives

**Definition.** Let  $I \subset \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  be a function, and  $p \in I^0$ . The function  $f$  is said to be **differentiable at  $p$**  if

$$\lim_{x \rightarrow p} \frac{f(x) - f(p)}{x - p} \text{ exists.}$$

If  $f$  is differentiable at  $p$  then we define the **derivative of  $f$  at  $p$** , denoted  $f'(p)$ , by

$$f'(p) := \lim_{x \rightarrow p} \frac{f(x) - f(p)}{x - p}$$

If  $S \subseteq I$ ,  $f$  is said to be **differentiable on  $S$**  if  $f$  is differentiable at  $x$  for all  $x \in S$ .

#### 5.1.

- (1) Let  $c \in \mathbb{R}$  be arbitrary and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = c$ . Prove that  $f$  is differentiable on  $\mathbb{R}$  and that  $f'(x) = 0$  for all  $x \in \mathbb{R}$ .
- (2) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x$ . Prove that  $f$  is differentiable on  $\mathbb{R}$  and that  $f'(x) = 1$ , for all  $x \in \mathbb{R}$ .
- (3) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^2 - x + 1$ . Prove that  $f$  is differentiable on  $\mathbb{R}$  and find  $f'(x)$  for  $x \in \mathbb{R}$ .
- (4) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = |x|$ . Prove that  $f$  is not differentiable at 0.

We can relate the notion of  $f$  being differentiable at  $p$  with our earlier notion of continuity.

**5.2.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  a function, and  $p \in I$ . Prove that  $f$  is differentiable at  $p$  if and only if there exists a function  $\phi : I \rightarrow \mathbb{R}$  that is continuous at  $p$  such that

$$f(x) = f(p) + (x - p)\phi(x).$$

Moreover, if there exists a function  $\phi : I \rightarrow \mathbb{R}$  that is continuous at  $p$  such that

$$f(x) = f(p) + (x - p)\phi(x),$$

then  $f'(p) = \phi(p)$ .

**Hint:** This is really just a very useful restatement of the definition.

Now many of our results about differentiability of  $f$  will follow from our rules for continuous functions, Problem 4.25, applied to  $\phi$ .

**5.3.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  a function, and  $p \in I$ . Prove that if  $f$  is differentiable at  $p$  then  $f$  is continuous at  $p$ .

In particular, using Problem 5.2, the usual rules of differentiation come from Problem 4.25 by some simple algebra.

**5.4.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$  a function,  $g : I \rightarrow \mathbb{R}$  a function,  $c \in \mathbb{R}$ , and  $p \in I$ . If  $f$  is differentiable at  $p$  and  $g$  is differentiable at  $p$  then

(1)  $c \cdot f$  is differentiable at  $p$  and

$$(c \cdot f)'(p) = c \cdot f'(p).$$

(2)  $f + g$  is differentiable at  $p$  and

$$(f + g)'(p) = f'(p) + g'(p).$$

(3)  $f \cdot g$  is differentiable at  $p$  and

$$(f \cdot g)'(p) = f(p) \cdot g'(p) + f'(p) \cdot g(p).$$

(4) if  $g(p) \neq 0$  then  $\frac{f}{g}$  is differentiable at  $p$  and

$$\left(\frac{f}{g}\right)'(p) = \frac{g(p) \cdot f'(p) - f(p) \cdot g'(p)}{(g(p))^2}.$$

**Hint:** You can prove these either directly from the definition or by writing  $f(x) = f(p) + (x - p) \cdot \phi(x)$  and  $g(x) = g(p) + (x - p) \cdot \psi(x)$  and using Problem 5.2. You should try both ways.

**5.5.** Let  $P(x) = a_0 + a_1x + \cdots + a_nx^n$  be a polynomial. Prove that for all  $x$

$$P'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}.$$

There is one more standard differentiation rule: the Chain Rule.

**5.6.** Let  $I, J \subseteq \mathbb{R}$  be intervals,  $f : I \rightarrow \mathbb{R}$  and  $g : J \rightarrow \mathbb{R}$  functions with  $f(I) \subseteq J$ . Let  $p \in I$ . Prove that if  $f$  is differentiable at  $p$  and  $g$  is differentiable at  $f(p)$  then  $g \circ f$  is differentiable at  $p$  and

$$(g \circ f)'(p) = g'(f(p)) \cdot f'(p).$$

**Hint:** Since  $g$  is differentiable at  $f(p)$  there exists a function  $\psi : J \rightarrow \mathbb{R}$  such that

$$g(x) = g(f(p)) + (x - f(p)) \cdot \psi(x)$$

for all  $x \in J$ . Replace  $x$  by  $f(x)$  and then use the fact that  $f(x) - f(p) = (x - p) \cdot \phi(x)$ .

Perhaps the most important applications of differentiation are in optimization and in estimation. In optimization we try to find the maximum or minimum value of a given function (often subject to one, or more, constraints).

## 2. The Mean Value Theorem

**Definition.** Let  $I \subseteq \mathbb{R}$  be an interval and  $f : I \rightarrow \mathbb{R}$  a function. We say  $p \in I$  is a **local maximum** if there exists a  $\delta > 0$  such that for all  $x \in I$  with  $|x - p| < \delta$ ,  $f(x) \leq f(p)$ . We say  $p \in I$  is a **local minimum** if there exists a  $\delta > 0$  such that for all  $x \in I$  with  $|x - p| < \delta$ ,  $f(x) \geq f(p)$ .

We begin with a lemma that relates local maxima and minima with differentiation.

**5.7.** Let  $I \subseteq \mathbb{R}$  be an interval and  $f : I \rightarrow \mathbb{R}$  a function. Prove that if  $p \in I^\circ$  is either a local maximum or a local minimum, and  $f$  is differentiable at  $p$ , then  $f'(p) = 0$ .

**Hint:** Assume  $p$  is a local maximum and compute the signs of

$$\lim_{x \rightarrow p^+} \frac{f(x) - f(p)}{x - p} \quad \text{and} \quad \lim_{x \rightarrow p^-} \frac{f(x) - f(p)}{x - p}.$$

If  $f$  is differentiable at  $p$  then these limits must be equal.

With this observation we can prove Rolle's Theorem. A version of the theorem was first stated by Indian astronomer Bhaskara in the 12th century however the first proof seems to be due to Michel Rolle in 1691.

**5.8.** Let  $a < b$ . Prove that if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous,  $f$  is differentiable on  $(a, b)$ , and  $f(a) = f(b)$ , then there exists a point  $p \in (a, b)$  with  $f'(p) = 0$ .

**Hint:** If  $f$  is not constant it has either a maximum value or a minimum value at some  $c \in (a, b)$ .

An immediate consequence of Rolle's Theorem is the very important Mean Value Theorem. The Mean Value Theorem is used extensively in estimation.

**5.9.** Let  $a < b$ . Prove that if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $f$  is differentiable on  $(a, b)$  then there exists  $c \in (a, b)$  with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Hint:** Construct a linear function  $l(x)$  with  $l(a) = f(a)$ ,  $l(b) = f(b)$  and consider  $g(x) = f(x) - l(x)$ .

We now give some consequences of the Mean Value Theorem.

**5.10.** Let  $a < b$ . Prove that if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous,  $f$  is differentiable on  $(a, b)$ , and  $f'(p) = 0$  for all  $p \in (a, b)$  then  $f(x)$  is a constant.

**5.11.** Let  $a < b$ ,  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, and  $f$  differentiable on  $(a, b)$ . Prove that

- (1) if  $f'(x) \geq 0$  for all  $x \in (a, b)$  then  $f$  is increasing on  $[a, b]$ , i.e. if  $a \leq x < y \leq b$ , then  $f(x) \leq f(y)$ .

- (2) if  $f'(x) \leq 0$  for all  $x \in (a, b)$  then  $f$  is decreasing on  $[a, b]$ , i.e. if  $a \leq x < y \leq b$ , then  $f(x) \geq f(y)$ .

## CHAPTER 6

# Integration

### 1. The Definition

Our final chapter is the other half of calculus, the definite integral. Again let's recall a familiar problem to motivate our definition.

**Problem.** Let  $f : [a, b] \rightarrow [0, \infty)$  be continuous. Find the area of the region bounded by  $x = a$ ,  $x = b$ ,  $y = f(x)$  and  $y = 0$ . Draw a few pictures of such  $f$ 's, e.g.,  $f(x) = x^2$  on  $[0, 2]$ . Geometry does not give us a formula for this area unless  $f$  is quite nice (e.g.,  $f(x) = 3$ ). Our approach for finding this area will be to use very thin rectangles to approximate the area and then use a limiting process to obtain the area. It looks complicated so be sure to draw some pictures to help you understand the notation.

In this chapter we define  $\int_a^b f(x) dx$ , commonly called the **(definite) Riemann integral** of  $f$  over  $[a, b]$ . You should recall from calculus the "short way" of computing this:

$$\int_1^2 x dx = \left. \frac{x^2}{2} \right|_1^2 = 2 - \frac{1}{2} = \frac{3}{2}.$$

This comes from the fundamental theorem of calculus, which we shall prove. We will define  $\int_a^b f(x) dx$  so that when  $f \geq 0$  and  $f$  is continuous, this number is the area of the region bounded by  $y = f(x)$ ,  $x = a$ ,  $x = b$  and the  $x$ -axis.

**Definition.** Let  $a < b$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function.

A **partition**  $P$  of  $[a, b]$  is an ordered finite set

$$a = x_0 < x_1 < \cdots < x_n = b.$$

If  $P$  and  $Q$  are two partitions of  $[a, b]$  we say that  $Q$  *refines*  $P$  if  $Q \supseteq P$ .

Let  $P = (x_0, x_1, \dots, x_n)$  be a partition of  $[a, b]$ . For  $1 \leq i \leq n$  we set

$$\begin{aligned} M_i(f, P) &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \\ &= \sup f([x_{i-1}, x_i]) \\ m_i(f, P) &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\} \\ &= \inf f([x_{i-1}, x_i]) \\ \Delta_i &= x_i - x_{i-1}. \end{aligned}$$

We define the **upper Riemann sum** of  $f$  with respect to  $P$ , denoted  $U(f, P)$ , by

$$U(f, P) = \sum_{i=1}^n M_i(f, P) \Delta_i$$

and we define the **lower Riemann sum** of  $f$  with respect to  $P$ , denoted  $L(f, P)$ , by

$$L(f, P) = \sum_{i=1}^n m_i(f, P) \Delta_i.$$

**6.1.** Let  $f(x) = x$  and  $g(x) = x^2$  for  $x \in [0, 1]$ . Let  $n \in \mathbb{N}$  and let  $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$  be a partition of  $[0, 1]$ . Draw a picture illustrating  $L(f, P_4)$ ,  $U(f, P_4)$ ,  $L(g, P_4)$ , and  $U(g, P_4)$ . Can you find expressions for  $L(f, P_n)$ ,  $U(f, P_n)$ ,  $L(g, P_n)$ , and  $U(g, P_n)$ ?

**6.2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded and let  $P$  be a partition of  $[a, b]$ . Let  $m = \inf f([a, b])$  and  $M = \sup f([a, b])$ .

(1) Prove

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a).$$

(2) Prove that if  $Q$  refines  $P$  then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

**Hint:** First prove that it suffices to show this when  $|Q| = |P| + 1$ . Then prove (2) in this case.

**Definition.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded. We define the **upper Riemann integral** of  $f$ , denoted  $U(f)$ , by

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

We define the **lower Riemann integral** of  $f$ , denoted  $L(f)$ , by

$$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

(Why do these exist?) We say  $f$  is **Riemann integrable** if  $L(f) = U(f)$ . In this case we call the common value the **(definite) Riemann integral of  $f$**  over the interval  $[a, b]$  which we denote  $\int_a^b f$ . Remember that the infimum or supremum of a set may not be a member of that set.

**6.3.** Show that for  $f$  and  $g$  as in Problem 6.1 for all  $n \in \mathbb{N}$ ,  $U(f, P_n) \neq U(f)$ ,  $L(f, P_n) \neq L(f)$  and similarly for  $g$ .

We begin by seeing that not every bounded function is integrable. Later we will prove that every continuous function is integrable.

**6.4.** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be given by

$$f(x) = \begin{cases} 0 & x \in [0, 1] \setminus \mathbb{Q} \\ 1 & x \in [0, 1] \cap \mathbb{Q}. \end{cases}$$

Prove, or disprove, the statement:  $f$  is integrable.

**6.5.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded and let  $P$  and  $Q$  be partitions of  $[a, b]$ . Prove that  $L(f, P) \leq U(f, Q)$ .

**Hint:** Consider the partition  $R = P \cup Q$ .

**6.6.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded. Show that  $L(f) \leq U(f)$ .

**6.7.** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be given by  $f(x) = x$  for all  $x \in [0, 1]$ . Let  $P_n$  be the partition of  $[0, 1]$  given by

$$P_n = \left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right).$$

- (1) Find  $L(f, P_n)$  and  $U(f, P_n)$ .
- (2) Find  $U(f, P_n) - L(f, P_n)$ .
- (3) Show  $f$  is integrable on  $[0, 1]$  and find  $\int_0^1 f$ .

**Remark.** Notice for *any* partitions  $P, Q$ , we have  $U(f, P) \neq L(f, Q)$ , even though  $U(f) = L(f)$ .

## 2. Integrable Functions

The definition of integrability can be difficult to use directly so we are fortunate to have this next problem.

**6.8.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded. Show that  $f$  is integrable on  $[a, b]$  if and only if for all  $\varepsilon > 0$  there exists a partition  $P = (x_0, \dots, x_n)$  of the interval  $[a, b]$  such that

$$U(f, P) - L(f, P) = \sum_{i=1}^n (M_i(f, P) - m_i(f, P)) \Delta_i < \varepsilon.$$

**6.9.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be an increasing function. Prove that  $f$  is integrable on  $[a, b]$ .

**Hint:** For an increasing function we know explicitly what  $M_i(f, P)$  and  $m_i(f, P)$  are.

**6.10.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Prove that  $f$  is integrable on  $[a, b]$ .

**Hint:** Use Problem 4.40 to conclude that  $f$  is uniformly continuous. Let  $\varepsilon > 0$  be arbitrary and choose  $\delta > 0$  so that if  $x, y \in [a, b]$  with  $|x - y| < \delta$  then  $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ . Let  $P$  be any partition with each  $\Delta_i x < \delta$  and use Problem 6.8.

**6.11.** Let  $f$  and  $g$  be integrable functions on  $[a, b]$ . Let  $c \in \mathbb{R}$  be an arbitrary constant. Prove that

- (1)  $c \cdot f$  is integrable on  $[a, b]$  and

$$\int_a^b c \cdot f = c \int_a^b f.$$

(2)  $f + g$  is integrable on  $[a, b]$  and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

**Hint:** Show

$$M_i(f + g, P) \leq M_i(f, P) + M_i(g, P)$$

and hence conclude that  $U(f + g, P) \leq U(f, P) + U(g, P)$ . Similarly, show that  $L(f + g, P) \geq L(f, P) + L(g, P)$ . Use Problem 6.8 to conclude integrability. Then prove the equation.

Look at  $f(x) = x$  and  $g(x) = 1 - x$  on the interval  $I = (0, 1)$ . Then  $\sup(f(I)) = 1 = \sup(g(I))$  and  $\inf(f(I)) = 0 = \inf(g(I))$  (notice that neither  $f$  nor  $g$  has a *maximum* or a *minimum* on  $I$ ). The function  $(f + g)(x) = f(x) + g(x) = x + 1 - x = 1$ , and is therefore constant on  $I$ . Thus,  $\sup((f + g)(I)) = 1 < 1 + 1 = \sup(f(I)) + \sup(g(I))$  and  $\inf((f + g)(I)) = 1 > 0 + 0 = \inf(f(I)) + \inf(g(I))$ .

**6.12.** Let  $f$  and  $g$  be integrable on  $[a, b]$  with  $f(x) \leq g(x)$  for all  $x \in [a, b]$ . Prove that  $\int_a^b f \leq \int_a^b g$ .

**6.13.** Let  $a < c < b$  and let  $f : [a, b] \rightarrow \mathbb{R}$ .

- (1) Assume  $f$  is integrable on  $[a, b]$ . Prove that  $f$  is integrable on  $[a, c]$  and  $[c, b]$ .
- (2) Assume that  $f$  is integrable on  $[a, c]$  and on  $[c, b]$ . Prove that  $f$  is integrable on  $[a, b]$  and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

**Definition.** If  $f$  is integrable on  $[a, b]$  we define  $\int_b^a f = -\int_a^b f$ . We define  $\int_a^a f = 0$ .

**6.14.** Let  $f$  be integrable on an interval containing  $a, b$  and  $c$ . Prove that, no matter what their order,

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

**6.15.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be integrable. Prove that

- (1)  $|f|$  is integrable on  $[a, b]$

**Hint:** Prove that  $M_i(|f|, P) - m_i(|f|, P) \leq M_i(f, P) - m_i(f, P)$ .

- (2)  $|\int_a^b f| \leq \int_a^b |f|$ .

**6.16.** Prove that if  $f$  and  $g$  are integrable on  $[a, b]$  then  $f \cdot g$  is integrable on  $[a, b]$ .

**Hint:** Since  $f$  and  $g$  are integrable we may define

$$L_f = \sup\{|f(x)| : a \leq x \leq b\},$$

$$L_g = \sup\{|g(x)| : a \leq x \leq b\}.$$

Use the fact that

$$f(x) \cdot g(x) - f(y) \cdot g(y) = f(x) \cdot (g(x) - g(y)) + (f(x) - f(y)) \cdot g(y)$$

to conclude

$$\begin{aligned} M_i(f \cdot g, P) - m_i(f \cdot g, P) \\ \leq L_f \cdot (M_i(g, P) - m_i(g, P)) + (M_i(f, P) - m_i(f, P)) \cdot L_g. \end{aligned}$$

Now use Problem 6.8

### 3. Fundamental Theorems of Calculus

Finally we reach the Fundamental Theorem of Calculus which relates integration and differentiation. It is commonly broken into two different theorems.

**6.17.** Prove the first Fundamental Theorem of Calculus:

Let  $f$  be integrable on  $[a, b]$ . Define a function  $F : [a, b] \rightarrow \mathbb{R}$  by

$$F(x) = \int_a^x f.$$

$F$  is uniformly continuous on  $[a, b]$ , and if  $f$  is continuous at  $c \in (a, b)$  then  $F'(c) = f(c)$ .

**Hint:** For uniform continuity show that if  $a \leq x \leq y \leq b$  then

$$F(y) - F(x) = \int_x^y f.$$

Now estimate  $\int_x^y f$  from above and below in terms of  $y - x$ . For  $F'(c) = f(c)$  show that

$$\frac{F(x) - F(c)}{x - c} - f(c) = \frac{1}{x - c} \int_c^x [f - f(c)].$$

Use the fact that  $f$  is continuous at  $c$  to conclude that the right hand side can be made arbitrarily small by taking  $x$  sufficiently close to  $c$ .

**6.18.** Prove the second Fundamental Theorem of Calculus:

If  $f$  is differentiable on  $[a, b]$  and  $f'$  is integrable on  $[a, b]$ , then  $\int_a^b f' = f(b) - f(a)$ .

**Hint:** If  $P = (x_0, x_1, \dots, x_n)$  is any partition of  $[a, b]$  then, by Problem 5.9, for each  $1 \leq i \leq n$  we can find  $t_i \in (x_{i-1}, x_i)$  so that

$$f(b) - f(a) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) = \sum_{i=1}^n f'(t_i) \Delta_i.$$

Show  $L(f', P) \leq f(b) - f(a) \leq U(f', P)$ .

#### 4. Integration Rules

Once we have the fundamental theorems of calculus we can prove two of the important rules of integration; integration by parts, and integration by substitution. These turn out to be reinterpretations of the product rule from differentiation and the chain rule respectively.

**6.19.** Prove that if  $f$  and  $g$  are functions that are differentiable on  $[a, b]$  and both  $f'$  and  $g'$  are integrable on  $[a, b]$  then

$$\int_a^b f(x) \cdot g'(x) = f(b) \cdot g(b) - f(a) \cdot g(a) - \int_a^b f'(x) \cdot g(x).$$

**Hint:** By Problem 5.4  $f \cdot g$  is differentiable and  $(f \cdot g)' = f' \cdot g + f \cdot g'$ .

**6.20.** Suppose that  $u$  is differentiable on  $[c, d]$  and  $u'$  is continuous on  $[c, d]$ . Let  $a = u(c)$  and  $b = u(d)$ . Suppose that  $f$  is continuous on  $u([c, d])$ . Prove that

$$\int_a^b f = \int_c^d (f \circ u) \cdot u'.$$

**Hint:** Let  $F(x) = \int_a^x f$ . Use Problem 6.17 to show that  $F$  is differentiable and use Problem 5.6 to show that  $F \circ u$  is differentiable. Now apply Problem 6.18.

**Remark.** Notice we don't need to assume that  $u([c, d]) = [a, b]$ .

## Appendix to Chapter 2

### 1. The Field Properties

Mathematicians study many different types of mathematical objects. You may have heard of groups, rings, topological spaces, smooth manifolds, vector spaces, Banach spaces, affine varieties, elliptic curves, etc. One of the objects which mathematicians study is called a *field*. In the introduction to the chapter, we mentioned several algebraic properties of  $\mathbb{R}$ . The crucial algebraic properties of  $\mathbb{R}$  can be summarized by saying that  $\mathbb{R}$  is a field. Notice that all the field properties (listed below) would certainly be demanded of any number system.

As we mentioned above, we will take all the properties on faith. Hence will call them axioms (in mathematics an axiom is a basic statement which is accepted without proof, for example the statement which says “There exists a set” is a basic axiom of mathematics).

AXIOM 1. There exists a set  $\mathbb{R}$ , which contains  $\mathbb{Q}$ . We may define two operations on  $\mathbb{R}$  called addition and multiplication, which extend normal addition and multiplication of rational numbers.

When we say that addition on  $\mathbb{R}$  extends addition on  $\mathbb{Q}$ , we mean that adding two real numbers which happen to be rational would be the same as the normal addition of rational numbers (and likewise for multiplication).

We will use all the standard notations regarding operations among numbers. For example  $a + b$  is the sum of  $a, b \in \mathbb{R}$ . As always, we write the symbol ‘=’ between two real numbers which are the same and the symbol ‘ $\neq$ ’ between two which are not.

AXIOM 2. Addition of real numbers is commutative: For every  $a, b \in \mathbb{R}$ ,  $a + b = b + a$ .

AXIOM 3. Addition of real numbers is associative: For every  $a, b, c \in \mathbb{R}$ ,  $a + (b + c) = (a + b) + c$ .

AXIOM 4. The real number zero is an additive identity: For each  $a \in \mathbb{R}$ ,  $0 + a = a$ .

AXIOM 5. Every real number has an additive inverse: For every  $a \in \mathbb{R}$ , there is a number  $b \in \mathbb{R}$  such that  $a + b = 0$ .

We mentioned above that there are many obvious facts about the real numbers that, strictly speaking, must be proven from the axioms. The following is an example (as are most of the exercises in this section).

**A2.1.** For every  $a \in \mathbb{R}$ , the additive inverse of  $a$  is unique. That is, if  $b$  and  $c$  are real numbers which satisfy  $a + b = 0$  and  $a + c = 0$ , we may conclude that  $b = c$ .

The previous problem justifies us saying THE additive inverse of  $a \in \mathbb{R}$  (rather than AN additive inverse). As usual, we will use the symbol  $-a$  for the additive inverse of  $a$ . Notice that, strictly speaking,  $-a$  is not the same symbol as  $(-1) \cdot a$  (that is the number negative 1 times the number  $a$ ). That the two symbols represent the same number will be one the obvious facts we prove below.

We can also now define subtraction: If  $a$  and  $b$  are natural numbers, then  $a - b$  is defined to be  $a + (-b)$  (in words  $a - b$  is the the sum of  $a$  and the additive inverse of  $b$ ).

AXIOM 6. Multiplication of real numbers is commutative: For every  $a, b \in \mathbb{R}$ ,  $ab = ba$ .

AXIOM 7. Multiplication of real numbers is associative: For every  $a, b, c \in \mathbb{R}$ ,  $a(bc) = (ab)c$ .

AXIOM 8. The number one is a multiplicative identity: For every  $a \in \mathbb{R}$ ,  $a \cdot 1 = a$ .

AXIOM 9. Every real number besides zero has a multiplicative inverse: For every  $a \in \mathbb{R}$ ,  $a \neq 0$ , there is a number  $b$  such that  $ab = 1$ .

We have a result about multiplicative inverses analogous to the one we had for additive inverse.

**A2.2.** For every  $a \in \mathbb{R}$  (other than zero), the multiplicative inverse of  $a$  is unique. That is, if  $b$  and  $c$  are real numbers which satisfy  $ab = 1$  and  $ac = 1$ , we may conclude that  $b = c$ .

Again we are now justified in referring to THE multiplicative inverse of  $a$ , which we will denote by  $a^{-1}$ . We define division in a similar manner as subtraction:  $a/b$  is defined to be  $ab^{-1}$  (that is,  $a/b$  is defined to be the product of  $a$  and the multiplicative inverse of  $b$ ).

AXIOM 10. Multiplication and addition satisfy the distributive property: For every  $a, b, c \in \mathbb{R}$ ,  $a(b + c) = ab + ac$ .

These algebraic properties of the real numbers are very important, but they are not unique to  $\mathbb{R}$ .  $\mathbb{Q}$  would satisfy all of these axioms and so is also a field. In general, there exist many different fields. The collection of complex numbers,  $\mathbb{C}$ , (with usual notions of addition and subtraction) is a field. For a prime number  $p$ , you may be familiar with the collection of numbers modulo  $p$ , often denoted  $\mathbb{Z}_p$ . It too is a field (and in contrast to  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  has finitely many elements). We will thus need more properties of  $\mathbb{R}$  to describe it uniquely.

The following (relatively simple) question might help you to better understand the axioms:

**A2.3.** Which axioms would still be satisfied if  $\mathbb{R}$  were replaced with  $\mathbb{Q}$ ? with  $\mathbb{Z}$ ? with  $\mathbb{N}$ ?

We will now give some more basic properties about the real numbers which follow from these axioms.

For our first result, we will see that the multiplication operation on  $\mathbb{R}$  still boils down to repeated addition (as long as one of the numbers is a natural number).

**A2.4.** Multiplication of real numbers by natural numbers is just repeated addition. That is, if  $a \in \mathbb{R}$ , and  $n \in \mathbb{N}$ ,  $na$  is the same as the number which results when  $a$  is added to itself  $n$  times.

**Hint:** Use induction.

As promised, we will show that the additive inverse of a real number is just that number, multiplied by  $-1$ .

**A2.5.** For every  $a \in \mathbb{R}$ , the product of  $a$  and  $-1$  is the additive inverse of  $a$ . That is,  $-a = (-1)a$ .

We can also define integral powers of real numbers in the usual way.

**Definition.** Let  $a \in \mathbb{R}$ . If  $n$  is a natural number, we define  $a^n$  to be the product of  $a$  with itself  $n$  times. If  $a^{-n}$  is defined to be the product of  $a^{-1}$  with itself  $n$  times. We also define  $a^0$  to be 1.

We have the following basic properties of powers:

**A2.6.** If  $a, b \in \mathbb{R}$  and  $m, n \in \mathbb{Z}$ , then

- (1)  $(a^m)^n = a^{mn} = (a^n)^m$ ,
- (2)  $a^{m+n} = a^m a^n$ , and
- (3)  $(ab)^m = a^m b^m$ .

**Hint:** These properties are by no means automatic. They must be proven, by careful reasoning, from the axioms.

## 2. The Order Properties

In the previous section we saw the algebraic (or field) properties of  $\mathbb{R}$ . In this one we will study the order properties. As mentioned in the introduction, a set is ordered if we have a rule which tells us, given two elements of the set, which is bigger.

**AXIOM 11.** The real numbers come equipped with an order which extends the order on  $\mathbb{Q}$ .

By ‘extends,’ we mean that if  $a$  and  $b$  are rational numbers, then  $a$  is less than  $b$  according to the order on  $\mathbb{Q}$  if and only if  $a$  is less than  $b$  according to the order on  $\mathbb{R}$ .

As usual, we denote the order by  $\leq$ .

AXIOM 12. The order is reflexive: For every  $a \in \mathbb{R}$ ,  $a \leq a$ .

AXIOM 13. The order is transitive: For every  $a, b, c \in \mathbb{R}$  such that  $a \leq b$  and  $b \leq c$ , we have  $a \leq c$ .

AXIOM 14. The order is antisymmetric: For every  $a, b \in \mathbb{R}$  such that  $a \leq b$  and  $b \leq a$ , we have  $a = b$ .

AXIOM 15. The order is a total order: For every  $a, b \in \mathbb{R}$ , either  $a \leq b$  or  $b \leq a$ .

$\mathbb{R}$  is by no means the only set that comes with an order. In fact, an order can be defined on any set (and many sets, like for example the set consisting of all the months in the year, have an obvious order). Actually, there are many different ways to define an order on  $\mathbb{R}$ , but there is only one order that will satisfy all the axioms we will list (and have listed).

We will also use the symbols  $<$ ,  $>$ , and  $\geq$  with their usual meanings (i.e.,  $a < b$  means  $a \leq b$  and  $a \neq b$ ). To make our words precise, we will pronounce  $a \leq b$  as “ $a$  is less than  $b$ ” and  $a < b$  as “ $a$  is strictly less than  $b$ ” (with similar phrasing for  $\geq$  and  $>$ ). Note then that “ $a$  is less than  $b$ ” includes the possibility that  $a = b$ . This is only a convention, but it is one used by many mathematicians.

Again we have many basic and obvious properties.

**A2.7.**  $\mathbb{R}$  satisfies the trichotomy property: if  $a, b \in \mathbb{R}$ , then exactly one of the following holds:

- (1)  $a < b$ ,
- (2)  $a > b$ , or
- (3)  $a = b$ .

As expected, a number which is strictly greater than zero is called **positive**, whereas a number which is either positive or zero (in other words a number that is greater than zero) is called **nonnegative**. We use the terms **negative** and **nonpositive** similarly (though nonpositive is typically used with less frequency).

### 3. The Ordered Field Properties

In this section, we will discuss how the algebraic (field) properties of  $\mathbb{R}$  interact with the order properties (again in ways that, if you think about them, should work in any system of numbers).

AXIOM 16. The order is preserved under addition by a fixed number: If  $a, b, c \in \mathbb{R}$  and  $a \leq b$  then  $a + c \leq b + c$ .

AXIOM 17. The product of two nonnegative numbers is again nonnegative: If  $a, b \in \mathbb{R}$ ,  $0 \leq a$ , and  $0 \leq b$  then  $0 \leq ab$ .

To say that  $\mathbb{R}$  satisfies these additional properties is to say that it is an *ordered field*. Notice that being an ordered field is much more restrictive than being a field and having an order. The field properties and the order properties must also interact in the right way (as described by the previous two axioms). For example, although there are many orders on the set of complex numbers,  $\mathbb{C}$ , there is no order which makes it into an ordered field (this is not too difficult to prove and we will do so below). Demanding that our numbers form an ordered field tells us that we cannot include imaginary numbers (or complex numbers) in our number system. It also turns out that there is no order on  $\mathbb{Z}_p$  which makes it into an ordered field.

Nevertheless,  $\mathbb{R}$  is not the only ordered field.  $\mathbb{Q}$  is an ordered field and there are many others. We will need one additional property, called the completeness axiom, to uniquely define  $\mathbb{R}$ . As we mentioned above, the completeness axiom is significantly deeper than the others and we will need to develop several new concepts in the next chapter before we can describe it.

Again we have many basic properties that follow from the axioms. As always, be careful not to use any facts other than the axioms (and other facts we have proven).

The next result shows that we may multiply inequalities by  $-1$  as long as we are willing to reverse the sign.

**A2.8.** Let  $a, b \in \mathbb{R}$ . If  $a \leq b$  then  $-b \leq -a$ .

More generally we may multiply an inequality by a real number, but, as expected, we must reverse the sign if the number is negative.

**A2.9.** Suppose  $a, b \in \mathbb{R}$  and  $a \leq b$ . If  $c \in \mathbb{R}$  is nonnegative, then  $ac \leq bc$ . If  $c$  is nonpositive then  $bc \leq ac$ .

Of course the same result holds for strict inequalities unless  $c = 0$  (by a similar proof).

**A2.10.** Given any number  $a \in \mathbb{R}$ , there is a number which is strictly larger.

**Hint:** Finding a number is not difficult, but prove rigorously that it is larger.

**A2.11.** If  $a \in \mathbb{R}$ ,  $a^2 \geq 0$  with  $a^2 = 0$  if and only if  $a = 0$ .

We will not have occasion to use the next two results, but they are of general interest and they provide insights into ordered fields.

**A2.12.** Suppose we have a field  $F$  (that is,  $F$  satisfies all the properties which we gave for  $\mathbb{R}$  in the section on the field properties). In addition suppose there is an element  $i \in F$  which satisfies  $i^2 = -1$ . Then there is no order on  $F$  which makes it into an ordered field (that is, no order which will satisfy all the properties given for  $\mathbb{R}$  in this chapter).

**Hint:** Suppose  $F$  does indeed satisfy all the properties we have given so far for  $\mathbb{R}$ . How does  $i$  compare to zero?

**A2.13.** There is no order on  $\mathbb{C}$  which makes it an ordered field.

This last result is important because  $\mathbb{C}$  does satisfy the completeness axiom. Thus there are fields other than  $\mathbb{R}$  which satisfy the completeness axiom, but no other ordered fields which satisfy it. Notice that  $\mathbb{Q}$  is an ordered field which is not complete and  $\mathbb{C}$  is a completed field which is not ordered.

All of the assumed properties of  $\mathbb{R}$  are now in place except completeness. The remaining statements of this chapter must therefore be proven from our axioms.

#### 4. Set Theory

We will not try and define what we mean by a **set**. Surprisingly this is actually quite complicated and there is a branch of mathematical logic called **set theory** that deals with this.

**Definition.** The basic set theory concepts we expect you to be familiar with are:

- (1)  $\emptyset$  is the **empty** set, i.e. the set with no elements.
- (2)  $A \subseteq B$  means that every element of  $A$  is an element of  $B$ , or for all  $x \in A$ ,  $x \in B$ . It is read “ $A$  is a **subset** of  $B$ .”
- (3)  $A = B$  means that  $A$  and  $B$  have the same elements. Another way of saying this is  $x \in A$  if and only if  $x \in B$ .
- (4)  $A \cap B = \{x : x \in A \text{ and } x \in B\}$ .  $A \cap B$  is read as “ $A$  intersect  $B$ ” and is called the **intersection** of  $A$  and  $B$ .
- (5)  $A \cup B = \{x : x \in A \text{ or } x \in B\}$ .  $A \cup B$  is read as “ $A$  union  $B$ ” and is called the **union** of  $A$  and  $B$ . If  $x \in A \cup B$  then  $x \in A$  or  $x \in B$ . It could be in both.
- (6)  $A$  and  $B$  are called disjoint sets if  $A \cap B = \emptyset$ .  $A$  and  $B$  are disjoint if they have no elements in common.

It turns out that the operations of union and intersection satisfy “distributive laws” reminiscent of those that hold for the real numbers:

**Theorem A2.14.** Suppose that  $A$ ,  $B$ , and  $C$  are sets. Then the following identities hold:

- (1)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- (2)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

**Definition.**

- (1)  $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$ .  $A \setminus B$  is read as “ $A$  minus  $B$ ” and is called the “set theoretic difference of  $A$  and  $B$ .”
- (2)  $A \Delta B = (A \setminus B) \cup (B \setminus A)$  and is called the **symmetric difference** of  $A$  and  $B$ .
- (3)  $A^c = \{x : x \notin A\}$  is called the **complement** of  $A$ .

**Caution:** To be honest  $A^c$  is not really a set since we have not said what  $x$  is other than it is not in  $A$ . When we use  $A^c$  we must have a universal set  $U$  in mind. The universal set is often unspecified and is simply inferred from the context. Then we can write  $A^c = U \setminus A$  which is unambiguous.

**Theorem A2.15.** Let  $A$  and  $B$  be sets. Then the following are true:

- (1)  $(A^c)^c = A$ .
- (2)  $(A \cap B)^c = A^c \cup B^c$ .
- (3)  $(A \cup B)^c = A^c \cap B^c$ .
- (4)  $A \triangle B = (A \cup B) \setminus (A \cap B)$ .

Statements (2) and (3) are called **De Morgan's Laws**.

**Definition.** If  $A$  and  $B$  are sets then the **Cartesian product** of  $A$  and  $B$  is defined to be

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

where  $(a, b)$  denotes the ordered pair.

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$  is the Cartesian plane.

**Definition.** We can define unions and intersections of large collections of sets. If  $I$  is a set, called the **index set**, and for all  $i \in I$ ,  $A_i$  is a set then we define

$$\bigcup_{i \in I} A_i = \{x : \text{there exists } i \in I \text{ such that } x \in A_i\}$$

$$\bigcap_{i \in I} A_i = \{x : \text{for all } i \in I, x \in A_i\}.$$

**Theorem A2.16.** If  $I$  is a set and for all  $i \in I$ ,  $A_i$  is a set, then

- (1)  $(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} A_i^c$ .
- (2)  $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$ .

These are the De Morgan's laws for unions and intersections over arbitrary index sets.

## 5. Cardinality

Having defined "function" we can proceed to some fun stuff concerning counting the number of elements in a set. How can you count an infinite set? How can you determine if two sets (even infinite ones) have the same number of elements, or as we shall say the same cardinality? Your first notions of size may be shattered. Read on.

**Definition.** If  $A$  and  $B$  are sets we write  $|A| = |B|$  if there exists a 1–1 onto function  $f : A \rightarrow B$ . We read  $|A| = |B|$  as "the cardinality of  $A$  equals the cardinality of  $B$ ."

**Note.** By Theorem 2.7 in Chapter 2,  $|A| = |B| \Rightarrow |B| = |A|$ .

Intuitively  $|A| = |B|$  means that both sets have the “same number of elements”. This is not startling for finite sets. It is no surprise that  $|\{a, b, c\}| = |\{1, 2, 3\}|$ . However this definition can lead to non-intuitive results. We can have  $A \subseteq B$ ,  $A \neq B$  yet  $|A| = |B|$  (how?).

**Definition.**  $A$  is *finite* if  $A = \emptyset$  or if there exists  $n \in \mathbb{N}$  with  $|A| = |\{1, 2, \dots, n\}|$ . (We then say  $|A| = 0$  or  $|A| = n$  accordingly.)  $A$  is *infinite* if  $A$  is not finite.  $A$  is *countably infinite* if  $|A| = |\mathbb{N}|$ .  $A$  is *countable* if  $A$  is finite or countably infinite.

Are all infinite sets also countably infinite?

**A2.17.** Prove that a set  $A$  is

- countably infinite if and only if we can write  $A = \{a_1, a_2, \dots\}$  where  $a_i \neq a_j$  if  $i \neq j$ .
- countably infinite if and only if  $A$  is infinite and we can write  $A = \{a_1, a_2, \dots\}$ .
- countable if and only if  $A = \emptyset$  or we can write  $A = \{a_1, a_2, \dots\}$ .

Deduce that if  $B \subseteq A$  and  $A$  is countable, then  $B$  is countable.

**A2.18.** Let  $|A| = |B|$  and  $|B| = |C|$ . Prove that  $|A| = |C|$ .

**A2.19.** Prove that

- $|\mathbb{N}| = |\{2, 4, 6, 8, \dots\}|$
- $|\mathbb{N}| = |\mathbb{Z}|$
- $|\mathbb{N}| = |\{x \in \mathbb{Q} : x > 0\}|$

**Hint:** Try to make an infinite list of all rationals in  $(0, 1]$ . Now try to make a list of all rationals  $> 1$ .

- $|\mathbb{N}| = |\mathbb{Q}|$ .

**A2.20.** If  $A$  is countable and  $B$  is countable prove that  $A \times B$  is countable.

**Hint:** You want to construct a list of all elements in  $A \times B$  (see 1.17). Can you make an infinite matrix of these elements starting with

$$\begin{array}{cccc} & a_1 & a_2 & a_3 & \dots \\ b_1 & \hline b_2 & \hline b_3 & \hline \vdots & \hline \end{array}$$

Can you take this matrix and make a list as in 1.17c)?

Our next problem is due to G. Cantor. It is a famous result which shook the mathematical world and has found its way into numerous “popular” math/science books. Cantor went insane. The problem’s solution relies on the decimal representation of a

real number. In turn this actually involves the notion of convergence of a sequence of reals which we address in chapter 3. But you can use it here. ( $1/3 = .333\dots$  means that  $1/3 = \lim_{n \rightarrow \infty} x_n$  where  $x_n = .33\dots3$  ( $n$  entries). Beware of this fact: Some numbers have 2 decimal representations, e.g.,  $1 = 1.000\dots = .999\dots$ . This can only happen to numbers which can be represented as decimals with 9 repeating forever from some point.)

**A2.21.** a) Prove that  $(0, 1)$  is not countable.

**Hint:** If it were countable then we can list  $(0, 1) = \{a_1, a_2, a_3, \dots\}$ . Write each  $a_i$  as a decimal to get an infinite matrix as the following example illustrates.

$$\begin{aligned} a_1 &= 0.13974\dots \\ a_2 &= 0.000002\dots \\ a_3 &= 0.55556\dots \\ a_4 &= 0.345587\dots \\ a_5 &= 0.9871236\dots \\ &\vdots \end{aligned}$$

Can you find a decimal in  $(0, 1)$  that is not on this list? Can you describe an algorithm for producing such a number? Could  $a = 0.5\dots$  be equal to  $a_1$ ? Could  $a = 0.54\dots$  be equal to  $a_1$  or  $a_2$ ?

b) Show that  $|(0, 1)| = |[0, 1]|$ .

c) If  $a < b$  show that  $|(0, 1)| = |(a, b)| = |[0, 1]| = |[a, b]|$ .

**Definition.**  $x \in \mathbb{R}$  is *irrational* if  $x \notin \mathbb{Q}$ . Thus  $\mathbb{R} \setminus \mathbb{Q}$  is the set of all irrational numbers.

Can you prove that irrationals exist? The next problem shows much more.

**A2.22.** a) Prove that if  $A$  and  $B$  are countable then  $A \cup B$  is countable.

b) Prove that  $\mathbb{R} \setminus \mathbb{Q}$  is *uncountable* (i.e., not countable).

c) Prove that if  $a < b$  then  $(a, b) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$ .

(This problem shows that every open interval contains irrationals. In the next chapter we will show that it contains rationals as well.)

d) Prove that if  $I$  is countable and for all  $i \in I$ ,  $A_i$  is a countable set then  $\bigcup_{i \in I} A_i$  is countable.

So irrationals do exist. Does this proof give you any explicit number in  $\mathbb{R} \setminus \mathbb{Q}$ ?

We have not defined  $|A| \leq |B|$  yet.

**A2.23.** Give a definition for  $|A| \leq |B|$ . Your definition should satisfy

- a)  $|A| \leq |A|$   
 b)  $|A| \leq |B|$  and  $|B| \leq |C|$  implies that  $|A| \leq |C|$ .

*Bonus.* Prove also that

- c)  $|A| \leq |B|$  and  $|B| \leq |A|$  implies that  $|A| = |B|$ .

**Definition.**  $|A| < |B|$  if  $|A| \leq |B|$  and  $|A| \neq |B|$ .

**A2.24.** Prove that for all sets  $A$ ,  $|A| < |\mathcal{P}(A)|$  (so no largest “cardinal number” exists). **Hint:** Show there does not exist a function  $f : A \rightarrow \mathcal{P}(A)$  which is onto by assuming such an  $f$  exists and considering  $B \in \mathcal{P}(A)$  where  $B = \{a \in A : a \notin f(a)\}$ .

**Note:**  $\mathcal{P}(A)$  denotes the set of all subsets of  $A$ . Thus  $\mathcal{P}(\{1, 2\}) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ .

## 6. Mathematical Induction

THE THEOREM OF MATHEMATICAL INDUCTION. *Let  $P(1), P(2), P(3), \dots$  be a list of statements, each of which is either true or false. Suppose that*

- i)  $P(1)$  is true  
 ii) For all  $n \in \mathbb{N}$ , if  $P(n)$  is true then  $P(n+1)$  is true.

*Then for all  $n \in \mathbb{N}$ ,  $P(n)$  is true.*

**A2.25.** Prove this theorem.

**Hint:** Suppose it were not true. Choose  $n_0$  to be the smallest integer so that  $P(n_0)$  is false.

**A2.26.** Use mathematical induction to establish the following. Make sure in your proof to precisely state what you are taking “ $P(n)$ ” to be.

- a) For all  $n \in \mathbb{N}$ ,  $1 + 2 + \dots + n = \frac{n(n+1)}{2}$   
 b) For all  $n \in \mathbb{N}$ ,  $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$   
 c) For all  $n \in \mathbb{N}$ , if  $n \geq 4$  then  $2^n < n!$ .

**Note:**  $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$ . This is called “ $n$  factorial.”

## Appendix to Chapter 3

### 1. Open and Closed Sets

Continuing our discussion of  $\mathbb{R}$  we turn to what is called “topology”. This is crucial material for our future discussion of limits and continuity.

**Definition.** Let  $\varepsilon > 0$ . The interval  $(a - \varepsilon, a + \varepsilon)$  is said to be an *open interval centered at  $a$  of radius  $\varepsilon$* .

**A3.1.** Let  $a < b$ . Show that  $(a, b)$  is an open interval of radius  $\varepsilon$  for some  $\varepsilon > 0$ . What is the center? What is  $\varepsilon$ ?

**Definitions.** Let  $S \subseteq \mathbb{R}$ .

- a)  $S$  is *open* if for all  $a \in S$  there exists  $\varepsilon > 0$  with  $(a - \varepsilon, a + \varepsilon) \subseteq S$
- b)  $S$  is *closed* if  $C(S) = \mathbb{R} \setminus S$  is open.

**QUICK QUESTION:** Is every  $S \subseteq \mathbb{R}$  either open or closed? Can you justify your answer?

**A3.2.** Prove that every open interval is an open set and every closed interval is a closed set.

**A3.3.** Classify as open, closed, both or neither

- a)  $\emptyset$    b)  $[0, 1]$    c)  $\mathbb{Q}$    d)  $\mathbb{R} \setminus \mathbb{Q}$    e)  $\mathbb{R}$    f)  $[0, 1] \cup [2, 3]$    g)  $\{\frac{1}{n} : n \in \mathbb{N}\}$

**Definitions.** Let  $S \subset \mathbb{R}$ .

- a)  $x \in \text{int}(S)$  if there exists  $\varepsilon > 0$  with  $(x - \varepsilon, x + \varepsilon) \subseteq S$ .
- b)  $x \in \text{bd}(S)$  if for all  $\varepsilon > 0$ ,  $(x - \varepsilon, x + \varepsilon) \cap S \neq \emptyset$  and  $(x - \varepsilon, x + \varepsilon) \cap C(S) \neq \emptyset$ .

**Note:** “int” is short for *interior* and “bd” is short for *boundary*.

**A3.4.** For each  $S$  find  $\text{int}(S)$  and  $\text{bd}(S)$

- a)  $[0, 1]$    b)  $(0, 1)$    c)  $\mathbb{Q}$    d)  $\mathbb{R}$    e)  $\{1, 2, 3\}$    f)  $\{\frac{1}{n} : n \in \mathbb{N}\}$

**A3.5.** Prove the following.  $S \subseteq \mathbb{R}$ .

- a)  $\text{int}(S) \subseteq S$  and  $\text{int}(S)$  is an open set.

- b)  $S$  is open  $\Leftrightarrow S = \text{int}(S)$ .  
 c)  $S$  is open  $\Leftrightarrow S \cap \text{bd}(S) = \emptyset$ .  
 d)  $S$  is closed  $\Leftrightarrow S \supseteq \text{bd}(S)$ .

**A3.6.** Prove the following

- a) If  $I$  is a set and for all  $i \in I$ ,  $A_i$  is an open set, then  $\bigcup_{i \in I} A_i$  is open.  
 b) If  $I$  is any set and for all  $i \in I$ ,  $F_i$  is a closed set then  $\bigcap_{i \in I} F_i$  is closed.  
 c) If  $n \in \mathbb{N}$  and  $A_i$  is an open set for each  $i \leq n$  then  $\bigcap_{i=1}^n A_i$  is open.  
 d) If  $n \in \mathbb{N}$  and  $A_i$  is a closed set for each  $i \leq n$  then  $\bigcup_{i=1}^n A_i$  is closed.

**A3.7.** Show by example that c) (and d)) in 2.17 cannot be extended to infinite intersections (unions).

**Definitions.** Let  $S \subseteq \mathbb{R}$ ,  $x \in \mathbb{R}$ .

- a)  $x$  is an *accumulation point* of  $S$  if for all  $\varepsilon > 0$ ,  $\{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\} \cap S \neq \emptyset$ .  
 b)  $S' = \{x : x \text{ is an accumulation point of } S\}$ .  
 c)  $x$  is an *isolated point* of  $S$  if  $x \in S \setminus S'$ .  
 d)  $\bar{S} = S \cup S'$ .

**Note:**  $\bar{S}$  is called the *closure* of  $S$ .

**A3.8.** Let  $S \subseteq \mathbb{R}$ . Prove the following.

- a)  $x \in S$  is an isolated point of  $S$  if and only if there exists  $\varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon) \cap S = \{x\}$ . Let  $S \subseteq \mathbb{R}$ .  
 b) Let  $x \in \mathbb{R}$ . Prove that  $x \in S'$  if and only if for all  $\varepsilon > 0$ ,  $(x - \varepsilon, x + \varepsilon) \cap S$  is infinite.

**A3.9.** For each set  $S$  below find  $S'$ ,  $\bar{S}$  and all isolated points of  $S$ .

- a)  $\mathbb{R}$    b)  $\emptyset$    c)  $\mathbb{Q}$    d)  $(0, 1]$    e)  $\mathbb{Q} \cap (0, 1)$    f)  $(\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1)$    g)  $\{\frac{1}{n} : n \in \mathbb{N}\}$

**A3.10.** Prove the following.  $S \subseteq \mathbb{R}$ .

- a)  $S$  is closed if and only if  $S \supseteq S'$ .  
 b)  $\bar{S}$  is closed.  
 c)  $S$  is closed if and only if  $S = \bar{S}$ .  
 d) If  $F \supseteq S$  and  $F$  is closed then  $F \supseteq \bar{S}$ .

## 2. Compactness

Our next topic in topology is compactness. The definition is quite abstract and will take effort to absorb. We will later prove that a continuous function on a compact domain achieves both a maximum and a minimum value — quite a useful thing in applications.

**Definitions.** Let  $S \subseteq \mathbb{R}$ .

- a) Let  $\{A_i\}_{i \in I}$  be a family of open sets.  $\{A_i\}_{i \in I}$  is an *open cover* for  $S$  if  $S \subseteq \bigcup_{i \in I} A_i$ .

For example,  $\{(n-1, n+1) : n \in \mathbb{Z}\}$  is an open cover of  $\mathbb{R}$ .

**Question.** For all  $x \in \mathbb{Q}$ , let  $\varepsilon_x > 0$ . Is  $\{(x - \varepsilon_x, x + \varepsilon_x) : x \in \mathbb{Q}\}$  necessarily an open cover of  $\mathbb{R}$ ?

- b) Let  $\{A_i\}_{i \in I}$  be an open cover for  $S$ . A *subcover* of this open cover is any collection  $\{A_i\}_{i \in I_0}$  where  $I_0 \subseteq I$  such that  $\bigcup_{i \in I_0} A_i \supseteq S$ .
- c)  $S$  is *compact* if every open cover of  $S$  admits a finite subcover, i.e., whenever  $\{A_i\}_{i \in I}$  is a family of open sets such that  $S \subseteq \bigcup_{i \in I} A_i$  then there exists a finite set  $F \subseteq I$  so that  $S \subseteq \bigcup_{i \in F} A_i$ .

This is a very abstract definition that requires study and time to absorb. Note that the definition requires that *every* open cover of  $S$  admits a finite subcover. To show  $S$  is not compact you only need construct *one* open cover without a finite subcover. Compactness plays a key role in analysis (and topology).

**A3.11.** Which of the following sets are compact?

- a)  $\{1, 2, 3\}$     b)  $\emptyset$     c)  $(0, 1)$     d)  $[0, 1)$     e)  $\mathbb{R}$

**A3.12.** Let  $S \subseteq \mathbb{R}$  be compact. Prove that

- a)  $S$  is bounded.                      b)  $S$  is closed.

**Hint:** Assume not in each case and produce an open cover without a finite subcover.

**A3.13.** Prove that  $[0, 1]$  is compact. Hint: Let  $\{A_i\}_{i \in I}$  be any open cover of  $[0, 1]$ . Let

$$B = \{x \in [0, 1] : [0, x] \text{ can be covered by a finite subcover of } \{A_i\}_{i \in I}\}.$$

Then  $0 \in B$  so  $B \neq \emptyset$ . Let  $x = \sup(B)$ . Show  $x \in B$ . Show  $x = 1$ .

**A3.14.** Let  $K \subseteq \mathbb{R}$  be compact and let  $F \subseteq K$  be closed. Prove that  $F$  is compact.

**Hint:** If  $\{A_i\}_{i \in I}$  covers  $F$  then  $\{A_i\}_{i \in I} \cup \{C(F)\}$  covers  $K$ .

**A3.15.** Let  $K \subseteq \mathbb{R}$  be closed and bounded.

- a) Prove  $\min(K)$  and  $\max(K)$  both exist if  $K \neq \emptyset$ .
- b) Prove that  $K$  is compact.

**Note:** From 3.12 and 3.15 we see that  $K \subseteq \mathbb{R}$  is compact  $\Leftrightarrow K$  is closed and bounded.

**A3.16.** Let  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$  be a nested sequence of closed, bounded and nonempty sets in  $\mathbb{R}$ . Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

**Hint:** Assume it is empty. Then

$$\mathbb{R} = C\left(\bigcap_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} C(I_n) \supseteq I_1.$$

**A3.17.** Let  $K \subseteq \mathbb{R}$  be compact and infinite. Prove that  $K' \neq \emptyset$ .

**Hint:** Assume  $K' = \emptyset$ .

**A3.18.** Let  $A \subseteq \mathbb{R}$  be bounded and infinite. Prove that  $A' \neq \emptyset$ .

### 3. Sequential Limits and Closed Sets

**Definition.** Let  $A \subseteq \mathbb{R}$ .  $A$  is *sequentially closed* if whenever  $(a_n)_{n=1}^{\infty}$  is a sequence in  $A$  converging to a limit  $a$ , then  $a \in A$ .

**A3.19.** If  $A \subseteq \mathbb{R}$  is closed then it is sequentially closed.

**A3.20.** If  $A \subseteq \mathbb{R}$  is sequentially closed then it is closed.

**A3.21.** If  $A \subseteq \mathbb{R}$  then  $A$  is closed if and only if it is sequentially closed.