Topology

M367K

Michael Starbird

June 5, 2008

Contents

Chapter 1

Cardinality and the Axiom of Choice

At the end of the nineteenth century, mathematicians embarked on a program whose aim was to axiomatize all of mathematics. That is, the goal was to emulate the format of Euclidean geometry in the sense of explicitly stating a collection of definitions and unproved axioms and then proving all mathematical theorems from those definitions and axioms. The foundation on which this program rested was the concept of a set. Axioms for set theory were proposed and then the goal was to cast known mathematical theorems in set theoretic terms. So the challenge for mathematicians was to take familiar objects, such as the real line, and familiar concepts, such as continuity and convergence, and recast them in terms of sets. From this effort arose the concept of a topological space and the field of topology.

We begin our exploration of set-theoretic topology by starting with perhaps the most basic mathematical idea—counting—and finding a way to generalize that notion to apply reasonably to infinite sets. We begin by exploring some of the most familiar infinite sets.

Definition (number sets)*.* Throughout these notes we will use the following notation:

1. $\mathbb{N}$ = the set of natural numbers (*i.e.*, the positive integers).

2. $\mathbb{Z}$ = the set of all integers.

3. $\mathbb{Q}$ = the set of rational numbers.

4. $\mathbb{R}$ = the set of real numbers.

The most basic counting question is deciding how to tell when two sets have the same size. Finite sets have the same size if they have the same number of elements. But 'the same number of elements' does not seem too useful a phrase when dealing with infinite sets, since there is no number that describes the size of an infinite set. Instead, we notice that if two finite sets have the same number of elements, then we can put the elements of one set in one-to-one correspondence with the elements of the other set. This idea of pairing up the elements of one set with the elements of the other is what we need to generalize the concept of size to infinite sets. The idea of a 1-to-1 function between two sets is the fundamental idea on which the exploration of the size or *cardinality* of infinite sets rests. So here then is the basic definition of two sets having the same cardinality.

Definition (cardinality). Two sets, A and B, have the *same cardinality* if and only if there exists a one-to-one, onto function $f : A \to B$. The cardinality of a set A is denoted $|A|$.

Definition (finite set, infinite set). A set X is *finite* if and only if it is empty or there is a 1–1, onto function $f : X \to \{1, 2, \ldots, n\}$ where n is an element of $\mathbb{N}$. A set that is not finite is *infinite*.

The cardinality of a finite set is simply the number of elements in that set: 0, 1, 2, 3, ...

A basic fact about the natural numbers $\mathbb{N}$, which you should feel free to use in your proofs, is that every non-empty set of natural numbers has a least element.

Theorem 1.1. The even positive integers have the same cardinality as the natural numbers.

Theorem 1.2. $|\mathbb{N}| = |\mathbb{Z}|$.

Theorem 1.3. Every subset of $\mathbb{N}$ is either finite or has the same cardinality as $\mathbb{N}$.

Definition (countable set). A set that has the same cardinality as a subset of $\mathbb{N}$ is *countable*.

So, a countable set is either finite or has the same cardinality as $\mathbb{N}$. The next theorem shows that the set of natural numbers is in some sense the smallest infinite set.

Theorem 1.4. Every infinite set has a countably infinite subset.

Theorem 1.5. A set is infinite if and only if there is a one-to-one function from the set into a proper subset of itself.

Theorem 1.6. $\mathbb{Q}$ is countable.

Theorem 1.7. The union of two countable sets is countable.

Theorem 1.8. The union of countably many countable sets is countable.

Theorem 1.9. The set of all finite subsets of a countable set is countable.

Exercise 1.10. Suppose a submarine is moving in a straight line at a constant speed in the plane such that at each hour, the submarine is at a lattice point. Suppose at each hour you can explode one depth charge at a lattice point that will kill the submarine if it is there. You do not know where the submarine is nor do you know where or when it started. Prove that you can explode depth charges in such a way that you will be guaranteed to eventually kill the submarine.

Definition (power set). For any set A, 2^A (or $\mathcal{P}(A)$) denotes the set of all subsets of A. (The empty set, denoted $\emptyset$, is a subset of any set.) 2^A is called the *power set* of A.

Exercise 1.11. Suppose $A = \{a, b, c\}$, then write down 2^A (or $\mathcal{P}(A)$), the power set of A.

Theorem 1.12. For any set A, there is a 1–1 function f from A into 2^A.

Theorem 1.13. For a set A, let P be the set of all functions from A to the two point set $\{0, 1\}$. Then $|P| = |2^A|$.

Theorem 1.14. There is a 1–1 correspondence between $2^{\mathbb{N}}$ and infinite sequences of 0's and 1's.

Theorem 1.15 (Cantor). There is no function from a set A onto 2^A.

Note that Cantor's Theorem implies that $2^{\mathbb{N}}$ is not a countable set. A set that is not countable is called uncountable. So $2^{\mathbb{N}}$ is an uncountable set. In fact, Cantor's theorem implies that there are infinitely many different infinite cardinal numbers:

Corollary 1.16. There are infinitely many different infinite cardinalities.

Theorem 1.17. There is a 1–1, onto function $f : [0, 1] \to [0, 1)$.

Theorem 1.18 (Schroeder-Bernstein). If A and B are sets such that there exist one-to-one functions f from A into B and g from B into A, then $|A| = |B|$.

Hint. We need to produce a 1–1, onto function $h : A \to B$. A useful picture is to depict A and B as parallel, equal length vertical lines and show f as a shrinking A into three-quarters of B and g shrinking B into three-quarters of A by drawing slanted lines between the top of A to the three-quarters point on B and vice versa, thinking of the bottom points going to one another under f and g. When defining h, for each point $x \in A$, either $h(x) = f(x)$ or $h(x) = g^{-1}(x)$. For some points x in A, you can not use g^{-1}, you must use f. Shade that interval, and shade its image under f. Now look at g of that interval, which in the picture is an interval in A. Could you use g^{-1} on those points? Why not? Continue the process and describe those points on which you must use f in your definition of h, and on which points you must use g^{-1}.

Theorem 1.19. $|\mathbb{R}| = |(0,1)| = |[0,1]|$.

Theorem 1.20. There is a 1–1 function from $\mathbb{R} \to 2^{\mathbb{N}}$.

Theorem 1.21. $|\mathbb{R}| = |2^{\mathbb{N}}|$.

1.1 *Zorn's Lemma, Axiom of Choice, Well-Ordering Principle

Three important statements in foundational mathematics are Zorn's Lemma, the Axiom of Choice, and the Well-Ordering Principle. These three statements are equivalent. They are accepted as fundamental axioms and used freely in most standard mathematics. We will use them in this course. In this section we give the relevant definitions and then state Zorn's Lemma, the Axiom of Choice, and the Well-Ordering Principle.

Definition (partially ordered set, poset). A set X is *partially ordered* by the relation $\mathcal{R}$ if and only if, for any elements x, y, and z in X,

1. $x\mathcal{R}x$,
2. if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$,
3. if $x\mathcal{R}y$ and $y\mathcal{R}x$ then $x = y$.

A partially ordered set is sometimes called a *poset*.

Example 1. For any set X, the power set of X, 2^X is a poset under the relation $\subseteq$.

Definition (least element). Let X be a poset with relation $\mathcal{R}$. Then an element a in X is a *least element* if and only if for any $x \in X$, $a\mathcal{R}x$.

Definition (maximal element). Let X be a poset with relation $\mathcal{R}$. Then an element m in X is a *maximal element* if and only if for any x in X, $m\mathcal{R}x$ implies that $m = x$.

Example 2. Recall we saw above that the power set 2^A is partially ordered by set inclusion. The set A is a maximal element, and, in fact, the only maximal element in this ordering.

Definition (totally ordered set). A poset is *totally ordered* if and only if it is partially ordered and every two elements are comparable (that is, for all x, y, either $x\mathcal{R}y$ or $y\mathcal{R}x$).

In general, we will use $\leq$ rather than $\mathcal{R}$ when talking about the relation in a totally ordered set.

Definition (well-ordered set). A set is *well-ordered* if and only if it is totally ordered and every non-empty subset has a least element.

The natural numbers are well-ordered.

Exercise 1.22. Show that the ordinary ordering on the reals is not a well-ordering.

Zorn's Lemma. Let X be a partially ordered set in which each totally ordered subset has an upper bound in X. Then X has a maximal element.

Axiom of Choice. Let $\{A_\alpha\}_{\alpha\in\lambda}$ be a collection of non-empty sets. Then there is a function $f : \lambda \to \bigcup_{\alpha\in\lambda} A_\alpha$ such that for each α in λ, $f(\alpha)$ is an element of A_α.

Well-ordering Principle. Every set can be well-ordered. That is, every set is in 1–1 correspondence with a well-ordered set.

1.2 *Ordinal numbers

In common English an ordinal number refers to the numerical position of an object: first, second, third, and so on. We will use the arabic numerals to denote the ordinal numbers with which we are most familiar: $0, 1, 2, 3, \ldots$. We can define ordinals in a manner that allows us to produce an ordered set of ordinals that includes infinite ordinals.

We start with the empty set, $\emptyset$. This set corresponds to 0.

The next ordinal, corresponding to 1, is the set containing the empty set, $\{\emptyset\}$. The next ordinal, corresponding to 2, is the set of its predecessors, namely $\{\emptyset, \{\emptyset\}\}$. The next ordinal, corresponding to 3, is the set of its predecessors, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Notice that each finite ordinal number k described in this way is a set containing k elements.

Continuing in this fashion, we can define each subsequent ordinal as the set of its predecessors.

For example, the first infinite ordinal, called ω_0, is the set of all the finite ordinals, namely, the set $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots\}$. The next ordinal is called $\omega_0 + 1$, then $\omega_0 + 2$, $\omega_0 + 3, \ldots$, then $2\omega_0, 2\omega_0 + 1, \ldots$; $\ldots k\omega_0, k\omega_0 + 1, k\omega_0 + 2, \ldots$, etc.

Note that every ordinal number has an immediate successor; however, not every ordinal has an immediate predecessor. For example, ω_0 has no immediate predecessor.

Note also that each ordinal is a set and, consequently, has a cardinality. The ordinal ω_0 is the first infinite ordinal and has the same cardinality as $\mathbb{N}$, denoted by $\aleph_0$, aleph nought. Each of the ordinal numbers $\omega_0 + 1$, $\omega_0 + 2$, $\omega_0 + 3$, and so on has countable cardinality, $\aleph_0$.

The first uncountable ordinal is called ω_1. It is the set of all the countable ordinals. Every ordinal preceding it, which is the same as in it, is countable.

The cardinality of ω_1 is less than or equal to the cardinality of $2^{\mathbb{N}}$ (or $\mathbb{R}$). However, the question of whether the cardinality of ω_1 is equal to the cardinality of $\mathbb{R}$ is the content of the Continuum Hypothesis. The Continuum Hypothesis asserts that these two cardinalities are in fact equal; however, it has been proved that the Continuum Hypothesis is independent of the standard Axioms of Set Theory (the Zermelo-Fraenkel axioms). That is, the Continuum Hypothesis can be neither proved nor disproved.

Continuum Hypothesis. The real numbers have the same cardinality as ω_1, the first uncountable ordinal.

Ordinal numbers are well-ordered by $\subseteq$, because the intersection of any set of ordinals is the smallest ordinal in the set, so every non-empty subset has a smallest element.

Theorem 1.23. Let $\{\alpha_i\}_{i\in\omega_0}$ be a countable set of ordinal numbers where each $\alpha_i < \omega_1$. Then there is an ordinal β such that $\alpha_i < \beta$ for each i and $\beta < \omega_1$.

Chapter 2

General Topology

In this chapter, we will start with the real number line and investigate some of its properties. We will then define a topological space as an abstraction of features of the real line. The topological ideas of limit point, convergence, open and closed sets, and continuity are all the result of capturing essential characteristics that we find in the real numbers.

2.1 The Real Number Line

We will not present an axiomatic definition of the real numbers. Instead, we will rely on our understanding of the real number line as the set of all decimal numbers ordered in their familiar way.

Let us first review the concepts necessary to define convergence of sequences and continuity of functions on the real number line.

Definition (open interval). In the real number line $\mathbb{R}$ define an *open interval* (a, b) as the set $\{x \in \mathbb{R} | a < x < b\}$.

Definition (open interval centered at x). Given $x \in \mathbb{R}$ and $\varepsilon > 0$ the *open interval centered at x of radius ε*, $B(x, \varepsilon)$ is the open interval $(x - \varepsilon, x + \varepsilon)$.

Definition (open set in $\mathbb{R}$). In $\mathbb{R}$ a set U is *open* if and only if for every point $x \in U$ there is an $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x + \varepsilon_x) \subseteq U$.

Theorem 2.1. The empty set is open and $\mathbb{R}$ is open.

Theorem 2.2. If U_1 and U_2 are open sets, then $U_1 \cap U_2$ is open. In fact, the intersection of finitely many open sets is open.

Theorem 2.3. The union of any collection of open sets in $\mathbb{R}$ is open.

Theorem 2.4. If U is open, then U is the union of open intervals.

Let us recall the definition of convergence of a sequence in $\mathbb{R}$:

Definition (convergent sequence). We say a sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}$ *converges* to x, or that x is the *limit of the sequence*, written as $x_i \to x$ if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x - x_i| < \varepsilon$ for all $i > N$.

We are in the process of recasting ideas from the real line in set-theoretic terms. So let's rephrase the definition of convergence *without* using distance, but, instead, in terms of open intervals. The definition above says that any open interval of radius ε centered at x contains *all but finitely many* of the elements of the sequence. Instead of restricting ourselves to open intervals centered at x, we can consider *any* open set containing x and define convergence as follows:

Definition (convergent sequence). If $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}$, then we say that x *converges* to x, written as $x_i \to x$ if and only if for any open set containing x, there is an $N \in \mathbb{N}$ such that $x_i \in U$ for all $i > N$.

If instead of studying sequences we wish to look at sets in $\mathbb{R}$ in general, we would like to define what we mean by saying that a point is "close" to (other) points in that set. One way is to abstract the concept of distance, but we don't actually need to have a distance, we can also use the (more general) concept of an open set. Let us see how it would play out in $\mathbb{R}$.

We could say that x is "close" to a set A if there is a sequence of elements of A that converges to x. If x is in A, however, we could always cheat and pick the constant sequence $\{x_i\}$ where $x_i = x$ for all (or almost all) i. This definition would not include the idea that x has nearby points from A. To avoid this problem we could say that x is "close" to a set A if there is a sequence $\{x_i\}$ of elements of $A - \{x\}$ that converges to x. But a more general definition would be to dispense with sequences altogether, and use the condition that the intersection of any open set containing x with $A - \{x\}$ is never empty. So we thus come to the following definition:

Definition (limit point in $\mathbb{R}$). Let A be a subset of $\mathbb{R}$ and x be a point in $\mathbb{R}$. Then x is a *limit point* of A if and only if for every open set U containing x $(U - \{x\}) \cap A)$ is not empty.

In other words, a limit point of a set is one that cannot be isolated from the rest of the set with an open set.

Definition (closed set in $\mathbb{R}$). A set in $\mathbb{R}$ is *closed* if and only if it contains all of its limit points.

Theorem 2.5. The intersection of any collection of closed sets in $\mathbb{R}$ is closed.

Theorem 2.6. The union of two closed sets in $\mathbb{R}$ is closed. In fact, the union of finitely many closed sets in $\mathbb{R}$ is closed.

Closed sets and open sets are related by the following theorem.

Theorem 2.7. A set in $\mathbb{R}$ is open if and only if its complement is closed.

Let us now review the definition of continuity that you probably first encountered in calculus—the ε-δ definition.

Definition (continuous function in $\mathbb{R}$). A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous at* x if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $z \in D$ and $|x - z| < \delta$ then $|f(x) - f(z)| < \varepsilon$. We say f is *continuous* if it is continuous for every point x in its domain D.

We wish to convert this definition into the language of open sets and intervals. It is saying that if a function is continuous in its domain, then if we pick an open interval $I = (f(x) - \varepsilon, f(x) + \varepsilon)$, then we can find an open interval $J = (x - \delta, x + \delta)$ that is mapped into I. So, in general, if we pick an open set $U \subseteq \mathbb{R}$ that contains a point $f(x)$, then we can find an open set $V \subseteq \mathbb{R}$ containing x whose image is contained inside U.

This set theoretic view of continuity allows us to re-word the concept of continuity in the language of open sets as follows:

Definition (continuous function in $\mathbb{R}$). A function $f : \mathbb{R} \to \mathbb{R}$ is a *continuous function* if and only if for every open set U in $\mathbb{R}$, $f^{-1}(U)$ is open in $\mathbb{R}$.

Theorem 2.8. The two definitions of continuity for real-valued functions on $\mathbb{R}$ are equivalent.

2.2 Open Sets and Topologies

We have now seen that several important concepts in analysis (convergence, limit points, closed sets, and continuity) can be defined using ideas about sets and their intersections and unions. In our familiar world of the real numbers, open sets were the central players in all these concepts. Now we would like to extend these concepts to spaces other than the familiar real numbers with their usual concept of open set. Our strategy is to abstract a more general concept of an open set from our experience with the real numbers. To that end, we isolate some of the conditions that were satisfied by the usual open sets of $\mathbb{R}$ and use those properties to define a *topology* and a *topological space*.

Definition (topology). Suppose X is a set. Then $\mathcal{T}$ is a *topology* for X if and only if $\mathcal{T}$ is a collection of subsets of X such that

1. $\emptyset \in \mathcal{T}$,

2. $X \in \mathcal{T}$,
3. if $U \in \mathcal{T}$ and $V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$,
4. if $\{U_\alpha\}_{\alpha \in \lambda}$ is any collection of sets each of which is in $\mathcal{T}$, then $\bigcup_{\alpha \in \lambda} U_\alpha \in \mathcal{T}$.

A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology for X.

If $(X, \mathcal{T})$ is a topological space, then $U \subseteq X$ is called an *open set* in $(X, \mathcal{T})$ if and only if $U \in \mathcal{T}$.

Theorem 2.9. Let $\{U_i\}_{i=1}^n$ be a *finite* collection of open sets in a topological space $(X, \mathcal{T})$. Then $\bigcap_{i=1}^n U_i$ is open.

Our first step toward understanding this abstract definition of a topological space is to confirm that the definition has captured relevant features of the prototype (that is, the real line) that spawned it. So our first example of a topological space will be the real number line where the collection of open sets in $\mathbb{R}$ that we talked about in the previous section is the standard topology on $\mathbb{R}$.

Example 3 (standard topology on $\mathbb{R}$). The *standard topology* $\mathcal{T}_{\text{std}}$ for $\mathbb{R}$ is defined as follows: a subset U of $\mathbb{R}$ belongs to $\mathcal{T}_{\text{std}}$ if and only if for each point p of U there is an open interval (a_p, b_p) such that $p \in (a_p, b_p) \subset U$.

Let us consider some other examples of topological spaces. Note that $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ are *different* topological spaces if $\mathcal{T} \neq \mathcal{T}'$, even though the underlying set X is the same. Keep in mind that open sets U are *elements* of the topology $\mathcal{T}$, and *subsets* of the space X. Elements of X, on the other hand are what we call the points of the space X.

Example 4 (discrete topology). For a set X, let 2^X be the set of all subsets of X. Then $\mathcal{T} = 2^X$ is called the *discrete topology* on X. The space $(X, 2^X)$ is called a *discrete topological space*.

Note the spelling: *discrete* topology, not *discreet* topology!

Example 5 (indiscrete topology). For a set X, $\mathcal{T} = \{\emptyset, X\}$ is called the *indiscrete topology* for X. So $(X, \{\emptyset, X\})$ is an indiscrete topological space.

Notice that the discrete topology has the maximum possible collection of open sets that any topology can have while the indiscrete topology has the minimum possible collection of open sets.

Example 6 (finite complement or co-finite topology). For any set X, the *finite complement (or co-finite) topology* for X is described as follows: a subset U of X is open if and only if $U = \emptyset$ or $X - U$ is finite.

Recall that a countable set is one that is either finite or countably infinite.

Example 7 (countable complement topology). For any set X, the *countable complement topology* for X is described as follows: a subset U of X is open if and only if $U = \emptyset$ or $X - U$ is countable.

Exercise 2.10. Verify that all the examples given above are indeed topologies; in other words, that they satisfy all four conditions needed to be a topology.

Exercise 2.11.

1. Describe some of the open sets you get if $\mathbb{R}$ is endowed with the topologies described above (standard, discrete, indiscrete, co-finite, and countable complement). Specifically, identify sets that demonstrate the differences among these topologies, that is, find sets that are open in some topologies but not in others.

2. For each of the topologies, determine if the interval $(0,1) \in \mathbb{R}$ is an open set in that topology.

We can generalize the standard topology on $\mathbb{R}$ to the Euclidean spaces $\mathbb{R}^n$. Rather than using open intervals to generate open sets, we use *open balls*:

Example 8. Let $\mathbb{R}^n$ be the set of all n-tuples of real numbers.

1. The *Euclidean distance* $d(x,y)$ between points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is given by the equation

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} .$$

2. The open ball of radius $\varepsilon > 0$ around point $p \in \mathbb{R}^n$ is the set $B(x,\varepsilon) = \{x \mid d(p,x) < \varepsilon\}$

3. A topology $\mathcal{T}$ for $\mathbb{R}^n$ is defined as follows: a subset U of $\mathbb{R}^n$ belongs to $\mathcal{T}$ if and only if for each point p of U there is a $\varepsilon_p > 0$ such that $B(x,\varepsilon_p) \subseteq U$.

This topology $\mathcal{T}$ is called the *standard topology for* $\mathbb{R}^n$.

Exercise 2.12. Give an example of a topological space and a collection of open sets in that topological space to show that the *infinite* intersection of open sets need not be open.

2.3 Limit Points and Closed Sets

As in $\mathbb{R}_{\text{std}}$, we will define the concept of a limit point using open sets, and then define closed sets as those sets that contain all their limit points.

Definition (limit point). Let $(X, \mathcal{T})$ be a topological space, A be a subset of X, and p be a point in X. Then p is a *limit point* of A if and only if for each open set U containing p, $(U - \{p\}) \cap A \neq \emptyset$. Notice that p may or may not belong to A.

In other words, p is a limit point of A if *all* open sets containing p intersect A at *some point other than* itself. Thus, the concept of a limit point gives us a way of capturing the idea of a point "being arbitrarily close" to a set *without* using the concept of distance. Instead we use the idea of open sets in a topology.

Definition (isolated point). Let $(X, \mathcal{T})$ be a topological space, A be a subset of X, and p be a point in X. If $p \in A$ but p is not a limit point of A, then p is an *isolated point* of A.

If p is an isolated point of A, then there is an open set U such that $U \cap A = \{p\}$.

Theorem 2.13. Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is *not* a limit point of A if and only if there exists an open set U with $p \in U$ and $U \cap A = \emptyset$.

Exercise 2.14. Give examples of a set A in a topological space and

1. a limit point of A that is an element of A;
2. a limit point of A that is not an element of A;
3. an isolated point of A;
4. a point not in A that is not a limit point of A.

Definition (closure of a set). Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$. Then the *closure* of A, denoted $\overline{A}$ or $\mathrm{Cl}(A)$, is A together with all of its limit points.

Definition (closed set). Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. A is *closed* if and only if $\mathrm{Cl}(A) = A$, in other words, if A contains all its limit points.

Theorem 2.15. For any topological space $(X, \mathcal{T})$ and $A \subseteq X$, $\overline{A}$ is closed, that is, for any set A in a topological space, $\overline{\overline{A}} = \overline{A}$.

A basic relationship between open sets and closed sets in a topological space is that they are complements of each other.

Theorem 2.16. Let $(X, \mathcal{T})$ be a topological space. Then the set A is closed if and only if $X - A$ is open.

Theorem 2.17. Let $(X, \mathcal{T})$ be a topological space, and let U be an open set and A be a closed subset of X. Then the set $U - A$ is open and the set $A - U$ is closed.

The properties of a topological space can be captured by focusing on closed sets instead of open sets. From that perspective, the four defining properties of a topological space are captured in the following theorem about closed sets.

Theorem 2.18. Let $(X, \mathcal{T})$ be a topological space:

i) $\emptyset$ is closed.

ii) X is closed.

iii) The union of finitely many closed sets is closed.

iv) Let $\{A_\alpha\}_{\alpha\in\lambda}$ be a collection of closed subsets in $(X, \mathcal{T})$. Then $\bigcap_{\alpha\in\lambda} A_\alpha$ is closed.

Exercise 2.19. Give an example to show that the union of infinitely many closed sets in a topological space may be a set that is not closed.

Exercise 2.20. Give examples of topological spaces and sets in them that:

1. are closed, but not open;
2. are open, but not closed;
3. are both open and closed;
4. are neither open nor closed.

Exercise 2.21. State whether each of the following sets are open, closed, both or neither.

1. In $\mathbb{Z}$ with the finite complement topology: $\{0, 1, 2\}$, $\{\text{prime numbers}\}$, $\{n \ : \ |n| \geq 10\}$.
2. In $\mathbb{R}$ with the standard topology: $(0, 1), (0, 1], [0, 1], \{0, 1\}, \{1/n \mid n \in \mathbb{N}\}$.

3. In $\mathbb{R}^2$ with the standard topology: $\{(x,y) \mid x^2+y^2=1\}$, $\{(x,y) \mid x^2+y^2>1\}$, $\{(x,y) \mid x^2+y^2 \geq 1\}$,
4. Which sets are closed in a set X with the discrete topology? indiscrete topology?

Theorem 2.22. For any set A in a topological space X, the closure of A equals the intersection of all closed sets containing A, that is,

$$\mathrm{Cl}(A) = \bigcap_{A \subseteq C,\ C \in \mathcal{C}} C$$

where $\mathcal{C}$ is the collection of all closed sets in X.

Informally, we can say $\overline{A}$ is the "smallest" closed set that contains A.

Exercise 2.23. Pick several different subsets of $\mathbb{R}$, and find their closure in:

1. the discrete topology;
2. the indiscrete topology;
3. the finite complement topology;
4. the standard topology.

Theorem 2.24. Let A, B be subsets of a topological space. Then

1. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$; and
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Exercise 2.25. In R^2 with the standard topology, describe the limit points and closure of the following two sets:

1. The topologist's sine curve:

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) | x \in (0,1) \right\}$$

2. The topologist's comb:

$$C = \{(x,0) | x \in [0,1]\} \cup \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, y \right) \mid y \in [0,1] \right\}$$

The following exercise is difficult.

Exercise 2.26. In the standard topology on $\mathbb{R}$, describe a non-empty subset C of the closed unit interval $[0,1]$ that is closed, contains no non-empty open interval, and where no point of C is an isolated point.

2.4 Interior, Exterior, and Boundary

Just as we have the concept of the "smallest" closed set containing A, we can considered what is the "largest" open set *contained in* A.

Definition (interior of a set). The *interior* of a set A in a topological space X, denoted $\overset{\circ}{A}$ or $\mathrm{Int}(A)$, is defined as:

$$\mathrm{Int}(A) = \bigcup_{U \subseteq A,\ U \in \mathcal{T}} U.$$

Exercise 2.27. Pick several different subsets of $\mathbb{R}$, and find their interior in:

1. the discrete topology;
2. the indiscrete topology;
3. the finite complement topology;
4. the standard topology.

We sometimes say, 'X is a topological space.' When we say that, we mean that there is a topology $\mathcal{T}$ on X that is implicit.

Theorem 2.28. Let $A \subseteq X$ be a subset of topological space X. Then $\mathrm{Int}(A)$ is the collection of points p such that there exists an open set U with $p \in U \subset A$.

Definition (boundary). The *boundary* of A, $\mathrm{Bd}(A)$ or ∂A, is $Cl(A) \cap Cl(X - A)$.

Finally, let us see how we would define the convergence of a sequence in a general topological space:

Definition (limit of a sequence). Given a sequence $\{x_i\}$ in a topological space X, we say x is a *limit of the sequence*, written as $x_i \to x$, if and only if for every open set U containing x, U contains all but finitely many of the x_i's. We also say x_i *converges* to x.

Theorem 2.29. Let $A \subseteq X$ be a subset of the topological space X. If $\{x_i\}_{i \in \mathbb{N}} \subset A$ and $x_i \to x$, then x is in the closure of A.

Question 2.30. Is the converse of the preceding theorem true? That is, if $A \subseteq X$ where X is a topological space and x is a limit point of A, then does there exist $\{x_i\}_{i \in \mathbb{N}} \subset A$ such that $x_i \to x$?

Exercise 2.31. Find an example of a topological space and a convergent sequence in that space, where the limit of the sequence is not unique.

Exercise 2.32. Consider sequences in $\mathbb{R}$ with the finite complement or cofinite topology. Which sequences converge? To what value(s) do they converge?

We will leave the study of continuity in general topological spaces for a later chapter. For now, we will focus on different ways of creating and describing topological spaces.

2.5 Bases

Because arbitrary unions of open sets are open, a topological space can have extremely complicated open sets. It is often convenient to describe a (simpler) subcollection of open sets that *generate*—in a prescribed way—all open sets in a given topology. So instead of having to specifically describe all of the open sets in a topological space $(X, \mathcal{T})$, we can more conveniently specify a subcollection, called a *basis* for the topology $\mathcal{T}$. Recall, for instance, that in order to define the open sets in the standard topology in $\mathbb{R}$ (respectively, $\mathbb{R}^n$) we used the concept of open intervals (respectively, open balls). We called a set U an open set if we could find an open interval (respectively, open ball) contained in U around every point in U. Thus, we could think of open sets as being made by taking arbitrary unions of these simpler open sets.

Definition (basis of a topology). Let $\mathcal{T}$ be a topology on a set X and let $\mathcal{B} \subseteq \mathcal{T}$. Then $\mathcal{B}$ is a *basis* for the topology $\mathcal{T}$ if and only if every element of $\mathcal{T}$ is the union of elements in $\mathcal{B}$. If $B \in \mathcal{B}$, we say B is a *basis element* or *basic open set*. Note that B is an *element* of the basis, but a *subset* of the space.

Theorem 2.33. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if

1. $\mathcal{B} \subseteq \mathcal{T}$,

2. $\emptyset \in \mathcal{B}$,

3. for each set U in $\mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subseteq U$.

Theorem 2.34. Let $\mathcal{B}_1 = \{(a, b) \subseteq \mathbb{R} \mid a,\ b \in \mathbb{Q}\}$, then $\mathcal{B}_1$ is a basis for the *standard topology* on $\mathbb{R}$. Let $\mathcal{B}_2 = \{(a, b) \cup (c, d) \subseteq \mathbb{R} \mid a,\ b,\ c,\ d$ are distinct irrational numbers$\}$, then $\mathcal{B}_2$ is also a basis for the *standard topology* on $\mathbb{R}$.

Suppose you are given a set X and a collection $\mathcal{B}$ of subsets of X. Under what circumstances is there a topology for which $\mathcal{B}$ is a basis? This question is answered in the following theorem. There is a subtle difference between the following theorem and the theorem two before this one. The former theorem started with a given topology and explored the question of when a collection of sets could form a basis for that particular topology. The following theorem explores the question of whether a given collection of sets could be a basis for *some topology* on X.

Theorem 2.35. Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for a topology for X if and only if the following conditions hold.

1. $\emptyset \in \mathcal{B}$,
2. for each point p in X there is a set U in $\mathcal{B}$ with $p \in U$, and
3. if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subseteq (U \cap V)$.

Theorem 2.35 allows us to describe topological spaces by first specifying a set X and then a collection $\mathcal{B}$ of subsets of X satisfying the 3 conditions listed in the theorem. Then the topology $\mathcal{T}$ with basis $\mathcal{B}$ is the collection of all possible unions of basis elements.

Example 9 (lower limit topology). We can define an alternative topology on $\mathbb{R}$, called the *lower limit topology*, generated by a basis consisting of all sets of the form $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$. Denote this space by $\mathbb{R}_{\text{LL}}$. The real line with the lower limit topology is sometimes called the *Sorgenfrey line* or $\mathbb{R}^1$(bad).

Theorem 2.36. Every open set in $\mathbb{R}$ with the standard topology is open in the lower limit topology, $\mathbb{R}_{\text{LL}}$.

Exercise 2.37. Show that the lower limit topology and the standard topology are different topologies on $\mathbb{R}$.

2.6 *Comparing Topologies

Definition (finer topology). Suppose a set X is given 2 topologies: $\mathcal{T}_1$ and $\mathcal{T}_2$. If $\mathcal{T}_1 \subset \mathcal{T}_2$ we say that $\mathcal{T}_2$ is a *finer* topology than $\mathcal{T}_1$ and that $\mathcal{T}_1$ is a *coarser* topology than $\mathcal{T}_2$. If $\mathcal{T}_1 \neq \mathcal{T}_2$ we would say *strictly* finer or coarser.

It's hard to remember which is the finer and which is the coarser topology when $\mathcal{T}_1 \subset \mathcal{T}_2$. A good way to remember which is which is that a comb with *more* teeth per inch is *finer* than one with fewer! In Exercises 2.36 and 2.37 we showed that the lower limit topology on $\mathbb{R}$ is finer than the standard topology on $\mathbb{R}$.

Note that it is possible that $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \not\subseteq \mathcal{T}_1$, in which case we say the topologies are *not comparable*.

Exercise 2.38. Give an example of two topologies in $\mathbb{R}$ that are not comparable.

2.7 Order Topology

Definition (order topology). Let X be a set totally ordered by $<$. Let $\mathcal{B}$ be the collection of all subsets of X of one of the following three forms:

$$\{x \in X \mid x < a\} \quad \{x \in X \mid a < x\} \quad \text{or} \quad \{x \in X \mid a < x < b\}.$$

Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X. The topology $\mathcal{T}$ is called the *order topology* for X.

Exercise 2.39. Show that the basis for the order topology described above is in fact a basis.

Theorem 2.40. The standard topology on $\mathbb{R}$ is the order topology given by the usual order.

Example 10 (lexicographically ordered square). Define an order relation on $[0,1] \times [0,1]$ by $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ or if $x_1 = x_2$ and $y_1 < y_2$. This order relation is called the *dictionary order* or *lexicographic order* and the corresponding order topology is called the *lexicographically ordered square.*

Exercise 2.41. In the lexicographically ordered square find the closures of the following subsets:

$$\begin{aligned}
A &= \left\{ \left(\frac{1}{n}, 0 \right) \mid n \in \mathbb{N} \right\} \\
B &= \left\{ \left(1 - \frac{1}{n}, \frac{1}{2} \right) \mid n \in \mathbb{N} \right\} \\
C &= \{(x, 0) \mid 0 < x < 1\} \\
D &= \left\{ \left(x, \frac{1}{2} \right) \mid 0 < x < 1 \right\} \\
E &= \left\{ \left(\frac{1}{2}, y \right) \mid 0 < y < 1 \right\}
\end{aligned}$$

Example 11. For each ordinal α, the collection of predecessors of α with the order topology form a space called α.

Theorem 2.42. Consider the topological space consisting of all ordinals less than ω_1, the first uncountable ordinal, with the order topology. Let $\{\alpha_i\}_{i\in\omega_0}$ be a countable set of distinct ordinal numbers where each $\alpha_i < \omega_1$. Then there is an ordinal $\beta < \omega_1$ that is a limit point of $\{\alpha_i\}_{i\in\omega_0}$.

Theorem 2.43. Consider the topological space consisting of all ordinals less than ω_1, the first uncountable ordinal, with the order topology. Let A and B be unbounded closed sets in this space. Then $A \cap B \neq \emptyset$.

2.8 Subspaces

If $(X, \mathcal{T})$ is a topological space and Y is a subset of X, then there is a natural topology that the topology $\mathcal{T}$ induces on Y:

Definition (subspace). Let $(X, \mathcal{T})$ be a topological space. For $Y \subseteq X$, the collection

$$\mathcal{T}_Y = \{U \mid U = V \cap Y \text{ for some } V \in \mathcal{T}\}$$

is a topology for Y called the *subspace topology*. The space $(Y, \mathcal{T}_Y)$ is called a (topological) *subspace* of X.

Theorem 2.44. The subspace topology $\mathcal{T}_Y$ is in fact a topology.

Question 2.45. In $Y = (0, 1)$, as a subspace of $\mathbb{R}_{\text{std}}$, is $[1/2, 1)$ closed, open, or neither?

Definition. The topology $\mathcal{T}_Y$ of Y from the definition of subspace is called the *relative topology* or *subspace topology*. The topological space $(Y, \mathcal{S})$ is a *subspace* of $(X, \mathcal{T})$ if and only if Y is a subset of X and $\mathcal{S}$ is the relative topology on Y.

Exercise 2.46. Consider a subspace $Y \subseteq (X, \mathcal{T})$. Is every subset $U \subseteq Y$ that is open with respect to the subspace topology also open in $(X, \mathcal{T})$?

Theorem 2.47. Let (Y, T_Y) be a subspace of $(X, \mathcal{T})$. A subset A is closed in (Y, T_Y) if and only if there is a set $B \subset X$, closed in X, such that $A = Y \cap B$.

Theorem 2.48. Let (Y, T_Y) be a subspace of $(X, \mathcal{T})$. A subset $A \in Y$ is closed in (Y, T_Y) if and only if $\text{Cl}_X(A) \cap Y = A$.

Theorem 2.49. Let $(X, \mathcal{T})$ be a topological space, and (Y, T_Y) be a subspace. If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$ is a basis for T_Y.

Exercise 2.50. Describe the relative topologies of the following subspaces of the lexicographically ordered square:

1. $D = \{(x, \frac{1}{2}) \mid 0 < x < 1\}$.
2. $E = \{(\frac{1}{2}, y) \mid 0 < y < 1\}$.

2.9 *Subbases

We saw in section 2.5 that it suffices to give a basis to specify a topology; all open sets are formed of arbitrary unions of basis elements. We can specify topologies in an even more condensed form by means of a *subbasis*, which allows arbitrary finite intersections of subbasis elements and then unions.

Definition (subbasis). Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a *subbasis* of $\mathcal{T}$ if and only if the collection $\mathcal{B}$ of all finite intersections of sets in $\mathcal{S}$ is a basis for $\mathcal{T}$.

An element of $\mathcal{S}$ is called a *subbasis element* or a *subbasic open set*.

Theorem 2.51. A basis for a topology is also a subbasis.

Theorem 2.52. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a subbasis for $\mathcal{T}$ if and only if

1. each element of $\mathcal{S}$ is in $\mathcal{T}$,
2. there is a finite collection $\{V_i\}_{i=1}^n$ of elements of $\mathcal{S}$ such that $\bigcap_{i=1}^n V_i = \emptyset$,
3. for each set U in $\mathcal{T}$ and point p in U there is a finite collection $\{V_i\}_{i=1}^n$ of elements of $\mathcal{S}$ such that
$$p \in \bigcap_{i=1}^{n} V_i \subseteq U \ .$$

Theorem 2.53. Let $\mathcal{S}$ be the following collection of subsets of $\mathbb{R}$: $\{x \mid x < a$ for some $a \in \mathbb{R}\}$ and $\{x \mid a < x$ for some $a \in \mathbb{R}\}$. Then $\mathcal{S}$ is a subbasis for $\mathbb{R}$ with the usual topology.

As with bases, we want to answer the question of when a given collection $\mathcal{S}$ of subsets of a set X is a subbasis for some topology on X.

Theorem 2.54. Let $\mathcal{S}$ be a collection of subsets of a set X. Then $\mathcal{S}$ is a subbasis for a topology on X if and only if every point of X is in some element of $\mathcal{S}$ and there are sets $\{U_i\}_{i=1}^n$ in $\mathcal{S}$ such that
$$\bigcap_{i=1}^{n} U_i = \emptyset \ .$$

The preceding theorem can thus be used to describe a topology by presenting a subbasis that generates it.

Chapter 3

Separation, Countability, and Covering Properties

At this point we know what a topology is, and we have a number of ways of describing a topology (*e.g.*, with a basis, with a total order, with a subbasis, with a topology on a larger space). We now will turn our attention to properties of these topologies.

At the end of this chapter, you will be asked to complete the following table. It makes sense to give it to you now so you can fill in the properties as we go.

Exercise 3.1. Make a grid with all our examples of topologies down the side. Across the top put each separation, countability, and covering property as we define it. Fill in squares indicating which examples have what properties.

3.1 Separation Properties

The first properties are the so-called *separation* properties, thus called because we use open sets to separate two points or closed sets from each other.

Definition (T_1, Hausdorff, regular, normal). Let $(X, \mathcal{T})$ be a topological space:

1. X is T_1 if and only if for all $x \in X$, $\{x\}$ is a closed set.
2. X is *Hausdorff* (or T_2) if and only if for each pair of points x, y in X, there are disjoint open sets U and V in $\mathcal{T}$ such that $x \in U$ and $y \in V$.
3. X is *regular* if and only if for each $x \in X$ and closed set A in X with $x \notin A$, there are open sets U, V such that $x \in U$, $A \subseteq V$ and

$U \cap V = \emptyset$.

4. X is *normal* if and only if for each pair of disjoint closed sets A and B in X, there are open sets U, V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

Theorem 3.2. Every Hausdorff space is T_1.

Theorem 3.3. Every regular, T_1 space is Hausdorff.

Theorem 3.4. Every normal, T_1 space is regular.

Exercise 3.5. Find (or define) a topological space that is not T_1.

Theorem 3.6. A topological space X is T_1 if and only if for any pair of distinct points x, y in X there are open sets $U \ni x$ and $V \ni y$ such that $x \notin V$ and $y \notin U$.

Theorem 3.7. A topological space X is regular if and only if for each point p in X and open set U containing p there is an open set V such that $p \in V$ and $\overline{V} \subseteq U$.

Theorem 3.8. A topological space X is normal if and only if for each closed set A in X and open set U containing A there is an open set V such that $A \subseteq V$, and $\overline{V} \subseteq U$.

Theorem 3.9. A topological space X is normal if and only if for each pair of disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subseteq U$, $B \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Exercise 3.10. Find two disjoint closed subsets A and B of a $\mathbb{R}^2$ with the standard topology such that $\inf\{d(a, b) \mid a \in A \text{ and } b \in B\} = 0$.

A natural question to ask is what properties carry through from a space to all of its subspaces:

Definition (hereditary property). Let P be a topological property (such as T_1, Hausdorff, etc.). A topological space X is *hereditarily* P if and only if for each subspace Y of X, Y has property P when Y is given the relative topology from X.

Theorem 3.11. A Hausdorff space is hereditarily Hausdorff.

Theorem 3.12. A regular space is hereditarily regular.

Theorem 3.13. Let A be a closed subset of a normal space X. Then A is normal when given the relative topology.

Normality Lemma 3.14. Let A and B be subsets of a topological space X and let $\{U_i\}_{i \in \mathbb{N}}$ and $\{V_i\}_{i \in \mathbb{N}}$ be two collections of open sets such that

1. $A \subseteq \bigcup_{i \in \mathbb{N}} U_i$,
2. $B \subseteq \bigcup_{i \in \mathbb{N}} V_i$,
3. for each i in $\mathbb{N}$, $\overline{U}_i \cap B = \emptyset$ and $\overline{V}_i \cap A = \emptyset$.

Then there are open sets U and V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

3.2 Countability Properties

We will now turn our attention to properties that have to do with countability. You may want to review Chapter 1 before moving ahead!

Definition (dense). Let A be a subset of a topological space X. Then A is *dense* in X if and only if $\overline{A} = X$.

Definition (separable). A space X is *separable* if and only if X has a countable dense subset.

Example 12. $\mathbb{R}_{\text{std}}$ is separable. Is $\mathbb{R}$ not separable in any of the topologies you've studied?

The choice of the word *separable* for the property described above is an unfortunate one, given that it is not related to the separability properties we described in the previous section.

Definition (2^{nd} countable). A space X is 2^{nd} *countable* if and only if X has a countable basis.

Definition (neighborhood basis). Let p be a point in a space X. A collection of open sets $\{U_\alpha\}_{\alpha \in \lambda}$ in X is a *neighborhood basis for* p if and only if $p \in U_\alpha$ for each $\alpha \in \lambda$ and for every open set U in X with p in U, there is an α in λ such that $U_\alpha \subseteq U$.

Definition (1^{st} countable). A space X is 1^{st} *countable* if and only if for each point x in X, there is a countable neighborhood basis for x.

Theorem 3.15. A 2^{nd} countable space is separable.

Theorem 3.16. A 2^{nd} countable space is 1^{st} countable.

Theorem 3.17. A 2^{nd} countable space is hereditarily 2^{nd} countable.

Theorem 3.18. If X is a separable, Hausdorff space, then $|X| \leq \left|2^{2^{\mathbb{N}}}\right|$.

Theorem 3.19. If $p \in X$ and p has a countable neighborhood basis, then p has a nested countable neighborhood basis.

Definition (convergence). Let $P = \{p_i\}_{i \in \mathbb{N}}$ be a sequence of points in a space X. Then the sequence P *converges to* a point x if and only if for every open set U containing x there is an integer M such that for each $m > M$, $p_m \in U$.

Theorem 3.20. Suppose x is a limit point of the set A in a 1^{st} countable space X. Then there is a sequence of points in A that converges to x.

Theorem 3.21. Every uncountable set in a 2^{nd} countable space has a limit point.

3.3 Covering Properties

The next properties we will study are the "covering" properties, so called because they involve collections of open sets that cover the space or a subset of the space.

Definition (cover, open cover). Let A be a subset of X and let $\mathcal{C} = \{C_\alpha\}_{\alpha \in \lambda}$ be a collection of subsets of X. Then $\mathcal{C}$ is a *cover of* A if and only if $A \subset \bigcup_{\alpha \in \lambda} C_\alpha$. $\mathcal{C}$ is an *open cover* if and only if each C_α is open.

Definition (compact). A space X is *compact* if and only if every open cover $\mathcal{C}$ of X has a finite subcover $\mathcal{C}'$. That is, $\mathcal{C}'$ is a finite open cover of X each of whose elements is a set in $\mathcal{C}$.

Definition (countably compact). A space X is *countably compact* if and only if every countable open cover of X has a finite subcover.

Definition (Lindelöf). A space X is *Lindelöf* if and only if every open cover of X has a countable subcover.

Theorem 3.22. Every countably compact and Lindelöf space is compact.

Theorem 3.23. Every 2^{nd} countable space is Lindelöf.

Theorem 3.24. Let A be a closed subspace of a compact (respectively, countably compact, Lindelöf) space. Then A is compact (respectively, countably compact, Lindelöf).

Theorem 3.25. Let $\mathcal{B}$ be a basis for a space X. Then X is compact (respectively, Lindelöf) if and only if every cover of X by basic open sets has a finite (respectively, countable) subcover.

Theorem 3.26. The closed subspace $[0,1]$ in the $\mathbb{R}_{\text{std}}$ topology is compact.

Theorem 3.27. Let A be a compact subspace of a Hausdorff space X. Then A is closed.

Heine-Borel Theorem 3.28. Let A be a subset of $\mathbb{R}^1$ with the standard topology. Then A is compact if and only if A is closed and bounded.

Theorem 3.29. If X is a Lindelöf space, then every uncountable subset of X has a limit point.

Theorem 3.30. Let X be a T_1 space. Then X is countably compact if and only if every infinite subset of X has a limit point.

Theorem 3.31. A compact, Hausdorff space is normal.

Theorem 3.32. A regular, Lindelöf space is normal.

3.4 Metric Spaces

The next category of topological spaces that we will consider are called metric spaces, so called because they rely on the idea of a distance between points. Metric spaces arise by considering the notion of the distance between two points in the familiar Euclidean spaces $\mathbb{R}^n$. The strategy is to look at that familiar idea of distance and cull from it central features, which then become the definition of a metric.

Definition (metric). A *metric* on a set M is a function $d : M \times M \to \mathbb{R}_+$, where $\mathbb{R}_+$ is the non-negative real numbers, such that for all $a, b, c \in M$:

1. $d(a, b) \geq 0$;
2. $d(a, b) = 0$ if and only if $a = b$;
3. $d(a, b) = d(b, a)$; and
4. $d(a, b) + d(b, c) \geq d(a, c)$,.

The last property is called the triangle inequality.

A space with a given metric has a very natural topology. In fact, we have used the concept of distance in $\mathbb{R}^1$ and $\mathbb{R}^n$ already to define the standard topologies on these spaces. Let us give the general process.

Theorem 3.33. Let d be a metric on the set X. Then the collection of all open balls $B(p, \epsilon) = \{y \in X | d(p, y) < \epsilon\}$ for every $p \in X$ and every $\epsilon > 0$ forms a basis for a topology on X. The topology for which it is a basis is called the *d-metric topology* for X.

Definition (metric space). A topological space $(X, \mathcal{T})$ is a *metric space* if and only if there is a metric d on X such that $\mathcal{T}$ is the d-metric topology. We will sometimes write a metric space as (X, d) to stand for X with the d-metric topology.

Example 13. The following are metric spaces (prove this!):
a) $\mathbb{R}^1$ with Euclidean metric $d(x, y) = |x - y|$.
b) $\mathbb{R}^2$ with Euclidean metric $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
c) $\mathbb{R}^2$ with the "taxi-cab" metric: $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.
d) Any set M, with the metric: $d(a, a) = 0$, and $d(a, b) = 1$ if $a \neq b$.
e) $\mathbb{Q}$ with the metric: $d(\frac{a}{b}, \frac{m}{n}) = max(|a-m|, |b-n|)$. (All fractions reduced.)

The same topology on a set may be generated by more than one metric. For example, show that the taxi-cab metric on $\mathbb{R}^2$ generates the standard topology on $\mathbb{R}^2$ just as the standard metric on $\mathbb{R}^2$ does.

Theorem 3.34. If X is a metric space and $Y \subset X$, then Y is a metric space.

Theorem 3.35. If $(X, \mathcal{T})$ is a metric space, then there is a metric d that generates $\mathcal{T}$ such that for each $x, y \in X$, $d(x, y) < 1$.

Theorem 3.36. If M is a metric space, then M is Hausdorff, regular, and normal.

Theorem 3.37. A separable metric space is second countable.

Theorem 3.38. In a metric space X, the following are equivalent:

1. X is 2^{nd} countable,
2. X is separable,
3. X is Lindelöf,
4. every uncountable set in X has a limit point.

Theorem 3.39. If a metric space is countably compact, it is compact.

Lebesgue Number Theorem. Let $\{U_\alpha\}_{\alpha \in \lambda}$ be an open cover of a compact set A in a metric space X. Then there exists a $\delta > 0$ such that for every point p in A, $B(p, \delta) \subseteq U_\alpha$ for some α.

A δ satisfying the theorem above is called a *Lebesgue number*.

Now you can finish the exercise that was assigned at the beginning of the chapter.

Exercise 3.40. Make a grid with all our examples of topologies down the side and all separation, countability, and covering properties across the top. Fill in squares indicating which examples have what properties.

3.5 *Further Countability Properties

Definition (Souslin property). A space X has the *Souslin property* if and only if X does *not* contain an uncountable collection of disjoint open sets.

Theorem 3.41. A separable space has the Souslin property.

Now that you know the definition of the Souslin property, you extend Theorem 3.38:

Theorem 3.42. In a metric space X, the following are equivalent:

1. X is 2^{nd} countable,
2. X is separable,

3. X is Lindelöf,
4. every uncountable set in X has a limit point,
5. X has the Souslin property,

3.6 *Further Covering Properties

The following uses the concept of a subbasis, as described in (optional) Section **??**:

Alexander Sub-basis Theorem 3.43. Let $\mathcal{S}$ be a subbasis for a space X. Then X is compact if and only if every subbasic open cover has a finite subcover. (A subbasic open cover is a cover of X each element of which is a set in the subbasis.)

Definition (locally finite). A collection $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$ of subsets of a space X is *locally finite* if and only if for each point p in X there is an open set U containing p such that U intersects only finitely many elements of $\mathcal{B}$.

Theorem 3.44. Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$ be a locally finite collection of subsets of a space X. Let C be a subset of λ. Then $\mathrm{Cl}(\bigcup_{\alpha\in C} B_\alpha) = \bigcup_{\alpha\in C} \overline{B}_\alpha$.

Example 14. Let $\mathcal{B} = \{[n, n+1] \subseteq \mathbb{R} \mid n \text{ is an integer}\}$. Then $\mathcal{B}$ is a locally finite collection in $\mathbb{R}_{\text{std}}$.

Definition (refinement of a cover). Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$ be a cover of X. Then $\mathcal{C} = \{C_\beta\}_{\beta\in\mu}$ is a *refinement* of $\mathcal{B}$ if and only if (i) $\mathcal{C}$ is a cover of X and (ii) for each $\beta \in \mu$ there is an $\alpha \in \lambda$ such that $C_\beta \subseteq B_\alpha$. The collection $\mathcal{C}$ is an *open refinement* if and only if each C_β is an open set.

Definition (paracompact). A space X is *paracompact* if and only if every open cover of X has a locally finite open refinement and X is Hausdorff.

Theorem 3.45. Every compact, Hausdorff space is paracompact.

Theorem 3.46. Let A be a closed subspace of a paracompact space. Then A is paracompact.

Theorem 3.47. A paracompact space is normal.

Theorem 3.48. A regular, T_1, Lindelöf space is paracompact.

Theorem 3.49. A metric space is paracompact.

3.7 *Properties on the ordinals

Theorem 3.50. ω_1 is countably compact but not compact.

Theorem 3.51. $\omega_1 + 1$ is compact.

Chapter 4

Maps Between Topological Spaces

Often in mathematics, once we have defined some sort of a mathematical structure, we then turn our attention to describing functions that acknowledge that structure. So now we turn our attention to functions between topological spaces where our goal is to decide what properties of the functions will describe a relationship between the topologies of the two spaces involved. Specifically, our first task is to decide what functions between topological spaces we want to call continuous. Also, we want to define what it means to say that two topological spaces are the "same".

4.1 Continuity

Recall our analysis of the definition of continuity of a real-valued function on the reals. By re-phrasing it in terms of open sets, we are able to create a definition of continuity that is meaningful for functions between any two topological spaces.

Definition (continuous function). Let X and Y be topological spaces. A function $f : X \to Y$ is a *continuous function* if and only if for every open set U in Y, $f^{-1}(U)$ is open in X. In this course, we will use the terms *map* and *continuous function* synonymously.

Theorem 4.1. Let $f : X \to Y$ be a function. Then the following are equivalent:

1. f is continuous,
2. for every closed set K in Y, $f^{-1}(K)$ is closed in X,

3. if p is a limit point of A in X, then $f(p)$ belongs to $\overline{f(A)}$.

To verify that our definition of continuity is a good one, let's verify that in the context of metric spaces, the traditional ε-δ definition of continuity is equivalent to the inverse images of open sets are open definition.

Theorem 4.2. If X and Y are metric spaces with metrics d_X and d_Y respectively, then a function $f : X \to Y$ is continuous if and only if for each point x in X and $\varepsilon > 0$, there is a $\delta > 0$ such that for each $y \in X$ with $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

When metric spaces are involved, continuity can be described in terms of convergence.

Theorem 4.3. Let X be a metric space and Y be a topological space. Then a function $f : X \to Y$ is continuous if and only if for each convergent sequence $x_n \to x$, $f(x_n)$ converges to $f(x)$.

For functions between metric spaces there is a stronger concept than continuity:

Definition (uniformly continuous). A function f from a metric space (X, d_X) to a metric space (Y, d_Y) is *uniformly continuous* if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for every x, $y \in X$, if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

Exercise 4.4. Give an example of a continuous function from $\mathbb{R}^1$ to $\mathbb{R}^1$ with the standard topology that is not uniformly continuous.

Theorem 4.5. Let $f : X \to Y$ be a continuous function from a compact metric space to a metric space Y. Then f is uniformly continuous.

Continuous functions preserve some of the topological properties we have studied.

Theorem 4.6. Let X be a compact (respectively, Lindelöf, countably compact) space and let $f : X \to Y$ be a continuous function that is onto. Then Y is compact (respectively, Lindelöf, countably compact).

Theorem 4.7. Let X be a separable space and let $f : X \to Y$ be a continuous, onto map. Then Y is separable.

There is a relationship between normality of a space X and the existence of some continuous functions from X into $[0, 1]$ with the standard topology. That important relationship is captured in a theorems known as Urysohn's Lemma and the Tietze Extension Theorem. The next lemma is used in the proof of Urysohn's Lemma.

Lemma 4.8. Let A and B be disjoint closed sets in a normal space X. Then for each diadic rational $r \in [0, 1]$ (r is a diadic rational if and only if it is of the form $q/2^k$ where q, k are integers) there exists an open set U_r such that $A \subseteq U_0$, $B \subseteq (X - U_1)$, and for $r < s$, $\overline{U_r} \subseteq U_s$.

Urysohn Lemma 4.9. A space X is normal if and only if for each pair of disjoint closed sets A and B in X, there exists a continuous function $f : X \to [0, 1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

Understanding the relationship between continuous functions and nested open sets allows us to prove the Tietze Extension Theorem below. Other proofs can be created that apply the statement of Urysohn's Lemma repeatedly to get a sequence of functions that converge to the desired function. But the proofs of the Tietze Extension Theorems are still difficult.

Tietze Extension Theorem 4.10. A space X is normal if and only if every continuous function f from a closed set A in X into $[0, 1]$ can be extended to a continuous function $F : X \to [0, 1]$. (F extends f means for each point x in A, $F(x) = f(x)$.)

Tietze Extension Theorem 4.11. A space X is normal if and only if every continuous function f from a closed set A in X into $(0, 1)$ can be extended to a continuous function $F : X \to (0, 1)$. (F extends f means for each point x in A, $F(x) = f(x)$.)

Definition (closed and open functions). A continuous function $f : X \to Y$ is *closed* if and only if for every closed set A in X, $f(A)$ is closed in Y. A continuous function $f : X \to Y$ is *open* if and only if for every open set U in X, $f(U)$ is open in Y.

Theorem 4.12. Let X be compact and Y be Hausdorff. Then any continuous function $f : X \to Y$ is closed.

4.2 Homeomorphisms

We now turn to the question of when two topological spaces are the "same."

Definition (homeomorphism). A function $f : X \to Y$ is a *homeomorphism* if and only if f is continuous, 1–1 and onto and $f^{-1} : Y \to X$ is also continuous.

Definition (homeomorphic spaces). X and Y, two topological spaces, are said to be *homeomorphic* if and only if there exists a homeomorphism $f : X \to Y$.

Theorem 4.13. For a continuous function $f : X \to Y$, the following are equivalent:

a) f is a homeomorphism.

b) f is 1–1, onto, and closed.

c) f is 1–1, onto, and open.

Theorem 4.14. For points $a < b$ in $\mathbb{R}^1$ with the standard topology, the interval (a, b) is homeomorphic to $\mathbb{R}^1$.

Theorem 4.15. Suppose $f : X \to Y$ is a 1–1 and onto continuous function, X is compact and Y is Hausdorff. Then f is a homeomorphism.

Theorem 4.16. Let $f : X \to Y$ be a function. Suppose $X = A \cup B$ where A and B are closed subsets of X. If $f \mid A$ is continuous and $f \mid B$ is continuous, then f is continuous.

The following statement cannot be proven without more rigorous definitions. In what sense could it be made rigorous? Is there a reasonable definition of a "topological property"?

Metatheorem 4.17. If X and Y are topological spaces and $f : X \to Y$ is a homeomorphism, then X and Y are the same as topological spaces, *i.e., any topological property of the space X is also a topological property of the space Y.*

4.3 Product Spaces

The Cartesian product of two topological spaces has a natural topology derived from the topologies on each one of the component spaces.

4.4 Finite Products

Definition (product). Let X and Y be two sets. The *product* $X \prod Y$, or *Cartesian product*, is the set of ordered pairs (x, y) where $x \in X$ and $y \in Y$.

Definition (product topology). Suppose X and Y are topological spaces. The *product topology* on the product $X \times Y$ is the topology whose basis is all sets of the form $U \times V$ where U is an open set in X and V is an open set in Y.

The Cartesian product of two topological spaces will be assumed to have the product topology unless otherwise specified.

Theorem 4.18. Let X and Y be topological spaces. The projection function $\pi_X : X \times Y \to X$ defined by $\pi_X((x,y)) = x$ is a continuous, open, onto map. Similarly, the projection function $\pi_Y : X \times Y \to Y$ defined by $\pi_Y((x,y)) = y$ is a continuous, open, onto map.

Theorem 4.19. Let X and Y be topological spaces. The projection functionπ_X : $X \times Y \to X$ need not be a closed map.

Theorem 4.20. Let X, Y, and Z be topological spaces. A function $g : Z \to X \times Y$ is continuous if and only if $\pi_X \circ g$ and $\pi_Y \circ g$ are both continuous.

Theorem 4.21. The space $\mathbb{R}^n$ is homeomorphic to $\prod_{i=1}^n \mathbb{R}_i$ where $\mathbb{R}_i = \mathbb{R}$.

Theorem 4.22. Let X and Y be Hausdorff spaces, then $X \times Y$ is Hausdorff. Let X and Y be regular spaces, then $X \times Y$ is regular.

Theorem 4.23. $\mathbb{R}_{\text{LL}}$ is normal, but $\mathbb{R}_{\text{LL}} \times \mathbb{R}_{\text{LL}}$ is not normal.

Theorem 4.24. Let X and Y be separable spaces, then $X \times Y$ is separable.

Theorem 4.25. Let X and Y be metric spaces, then $X \times Y$ is metric.

Theorem 4.26. Let X and Y be compact spaces, then $X \times Y$ is compact.

4.5 *Infinite Products

Definition (infinite product). Let $\{X_\alpha\}_{\alpha \in \lambda}$ be a collection of spaces. The *product* $\prod_{\alpha \in \lambda} X_\alpha$, or *Cartesian product*, is a generalization of the familiar n-tuples. Define $\prod_{\alpha \in \lambda} X_\alpha$ to be the set of functions $\{f : \lambda \to \bigcup_{\alpha \in \lambda} X_\alpha \mid \forall \alpha f(\alpha) \in X_\alpha\}$.

For $f \in \prod_{\alpha \in \lambda} X_\alpha$, $f(\alpha)$ is the α^{th} *coordinate* of f. We often write f as $\{f_\alpha\}_{\alpha \in \lambda}$ where $f(\alpha) = f_\alpha$.

Definition (product topology). For each β in λ, define the projection function $\pi_\beta : \prod_{\alpha \in \lambda} X_\alpha \to X_\beta$ by $\pi_\beta(f) = f(\beta)$. We define a topology on $\prod_{\alpha \in \lambda} X_\alpha$ to make the projection functions continuous. That is, define the *product topology* on $\prod_{\alpha \in \lambda} X_\alpha$ to be the one generated by the sub-basis of elements of the form $\pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β.

Theorem 4.27. Let $\prod_{\alpha \in \lambda} X_\alpha$ be the product of topological spaces $\{X_\alpha\}_{\alpha \in \lambda}$. The function $\pi_\beta : \prod_{\alpha \in \lambda} X_\alpha \to X_\beta$ is a continuous, open, onto map.

Theorem 4.28. Let $\prod_{\alpha \in \lambda} X_\alpha$ be the product of topological spaces $\{X_\alpha\}_{\alpha \in \lambda}$. A function $g : Y \to \prod_{\alpha \in \lambda} X_\alpha$ is continuous if and only if $\pi_\beta \circ g$ is continuous for each β in λ.

Theorem 4.29. Let $\{X_\beta\}_{\beta \in \mu}$ be a collection of Hausdorff (resp. regular) spaces. Then $\prod_{\beta \in \mu} X_\beta$ is Hausdorff (resp. regular).

Theorem 4.30. Let $\{X_\beta\}_{\beta\in\mu}$ be a collection of separable spaces where $|\mu| \leq 2^{\omega_0}$, then $\prod_{\beta\in\mu} X_\beta$ is separable.

Theorem 4.31. Let $\{X_\beta\}_{\beta\in\mu}$ be a collection of separable spaces. Then $\prod_{\beta\in\mu} X_\beta$ has the Souslin property.

Theorem 4.32. Let $\{X_i\}_{i\in\omega}$ be a countable collection of metric spaces. Then $\prod_{i\in\omega} X_i$ is a metric space.

Theorem 4.33. The Cantor set is the product $\prod_{n\in\mathbb{N}}\{0,1\}$ where $\{0,1\}$ has the discrete topology.

Tychonoff Theorem 4.34. Let $\{X_\beta\}_{\beta\in\mu}$ be a collection of compact spaces. Then $\prod_{\beta\in\mu} X_\beta$ is compact.

Definition (completely regular). A space X is *completely regular* if and only if for each point p and open set U with $p \in U$, there is a continuous function $f : X \to [0,1]$ such that $f(p) = 0$ and $f(X - U) = 1$.

Theorem 4.35. Let X be a completely regular, T_1 space. Then X can be embedded in a product of $[0,1]$'s.

Theorem 4.36. For any separable space X, the topological space 2^X has the Souslin property.

Theorem 4.37. The space $2^{\mathbb{R}}$ is separable.

Chapter 5

Connectedness

We now go back to studying properties of topological spaces. In this chapter we will study how to capture the idea of a space being connected.

Before we continue, you should consider the following questions, and come up with your own answers. Do *not* read ahead until you have answered the questions!

Question 5.1. Consider the spaces $Y = (0,1) \cup (2,3)$ and $Z = (4,5) \cup (5,6) \subset \mathbb{R}$ We have the intuitive understanding that Y and even Z are each made up of two pieces that don't hang together, or connect. Come up with at least one definition of what it might mean to be connected.

Question 5.2. Now consider the following two sets in $\mathbb{R}^2$ with the standard topology: the topologist's sine curve and the topologist's comb. Are their closures connected or not in your definition(s)? Recall that the topologist's sine curve is defined by:

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) | x \in (0,1) \right\}$$

and the topologist's comb is defined by:

$$C = \{(x,0) | x \in [0,1]\} \cup \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, y\right) \mid y \in [0,1] \right\}$$

OK, now you can read ahead....

5.1 Connectedness

There are several different, non-equivalent ways of defining connectedness. Perhaps the most natural one is the idea of being able to "walk" from any point to another without leaving the space. This concept, called path or arc-wise connectivity, will be studied in the next section. For now, we will focus on the observation we made above, that if a set can be split up into two non-empty disjoint open sets, then it is *not* connected.

Definition (connectedness). Let X be a topological space. A space X is *connected* if and only if X is not the union of two non-empty, disjoint open sets.

Definition (separated sets). Let X be a topological space. Subsets A, B of X are *separated* if and only if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. The notation $X = A \mid B$ means $X = A \cup B$ and A and B are separated sets.

Theorem 5.3. The following are equivalent:

1. X is connected;
2. there is no continuous function $f : X \to \mathbb{R}$ such that $f(X) = \{0, 1\}$;
3. X is not the union of two non-empty separated sets;
4. X is not the union of two non-empty, disjoint closed sets.

To prove that R^1 with the standard topology is connected, we need to use the least upper bound property of the reals, or something equivalent.

Theorem 5.4. The space $\mathbb{R}^1$ with the standard topology is connected.

Theorem 5.5. Let A, B be separated subsets of a space X. If C is a connected subset of $A \cup B$, then $C \subseteq A$, or $C \subseteq B$.

Theorem 5.6. Let C be a connected subset of X. If D is a subset of X so that $C \subseteq D \subseteq \overline{C}$, then D is connected.

Exercise 5.7. Consider the closure (in the usual topology on $\mathbb{R}^2$) of the topologist's sine curve. Is it connected?

Theorem 5.8. Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of connected subsets of X and E be another connected subset of X such that for each α in λ, $E \cap C_\alpha \neq \emptyset$. Then $E \bigcup \big(\bigcup_{\alpha\in\lambda} C_\alpha\big)$ is connected.

Theorem 5.9. Let $f : X \to Y$ be a continuous, onto function. If X is connected, then Y is connected.

Theorem 5.10. For spaces X and Y, $X \times Y$ is connected if and only if each of X and Y is connected.

Theorem 5.11. For spaces $\{X_\alpha\}_{\alpha\in\lambda}$, $\prod_{\alpha\in\lambda} X_\alpha$ is connected if and only if for each α in λ, X_α is connected.

Theorem 5.12. Let A be a countable subset of $\mathbb{R}^n$ $(n \geq 2)$. Then $\mathbb{R}^n - A$ is connected.

Theorem 5.13. Let X be a countable, regular, T_1 space. Then X is not connected.

Theorem 5.14. Let X be a connected space, C a connected subset of X, and $X - C = A \mid B$. Then $A \cup C$ and $B \cup C$ are each connected.

Definition (component). Let X be a space and $p \in X$. The *component* or *connected component* of p in X is the union of all connected subsets of X that contain p.

Theorem 5.15. Each component of X is connected and closed.

Theorem 5.16. Let A and B be closed subsets of a compact,Hausdorff space X such that no component intersects both A and B. Then $X = H \mid K$ where $A \subseteq H$ and $B \subseteq K$.

Example 15. This example will demonstrate the necessity of the "compactness" hypothesis of Theorem 5.16. Let X be the subset of $\mathbb{R}^2$ equal to $([0,1] \times \bigcup_{i\in\omega_0}\{1/i\}) \bigcup \{(0,0),(1,0)\}$ with the subspace topology. Show that the conclusion to Theorem 5.16 fails when $A = \{(0,0)\}$ and $B = \{(1,0)\}$.

5.2 Continua

Definition (continuum). A *continuum* is a connected, compact, Hausdorff space.

Theorem 5.17. Let U be a proper, open subset of a continuum X. Then each component of $\overline{U}$ contains a point of $\operatorname{Bd} U$. (Recall: $\operatorname{Bd} U = \overline{U} - U$.)

Theorem 5.18. ("To the boundary" theorem.) Let U be a proper, open subset of a continuum X. Then each component of U has a limit point on $\operatorname{Bd} U$.

Theorem 5.19. No continuum X is the union of a countable number (> 1) of disjoint closed subsets.

Example 16. This example shows the necessity of the compactness hypothesis on X.

The example X pictured above is a subset of the plane which is the union of a countable number of arcs as shown. Show that X is connected.

Theorem 5.20. Let $\{C_i\}_{i\in\omega}$ be a collection of continua such that for each i, $C_{i+1} \subseteq C_i$. Then $\bigcap_{i\in\omega} C_i$ is a continuum.

Theorem 5.21. Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of continua indexed by a well-ordered set λ such that if $\alpha < \beta$, then $C_\beta \subseteq C_\alpha$. Then $\bigcap_{\alpha\in\lambda} C_\alpha$ is a continuum.

Definition (separating and non-separating points). Let X be a connected set. A point p in X is a *non-separating point* if and only if $X - \{p\}$ is connected. Otherwise p is a *separating* point.

Theorem 5.22. Let X be a continuum, p be a point of X, and $X - \{p\} = H \mid K$. Then $H \cup \{p\}$ is a continuum and if $q \neq p$ is a non-separating point of $H \cup \{p\}$, then q is a non-separating point of X.

Theorem 5.23. Let X be a metric continuum. Then X has at least two non-separating points.

Theorem 5.24. Let X be a continuum. Then X has at least two non-separating points.

Theorem 5.25. Let X be a metric continuum with exactly two non-separating points. Then X is homeomorphic to $[0, 1]$.

Theorem 5.26. Let X be a metric continuum with more than one point where no point separates but every pair of points separates. Then X is homeomorphic to $\mathbb{S}^1$.

5.3 Path or Arc-Wise Connectedness

Perhaps one of the ways you thought of defining connectedness at the beginning of the chapter was the idea of being able to "walk" from any point of the set to any other without leaving the set. This idea leads to the property called *path* or *arc-wise connectedness*.

Definition (path or arc-wise connected). A space X is *arc-wise connected* or *path connected* if and only if for each pair of points p, $q \in X$ there is an embedding $h : [0,1] \to X$ such that $h(0) = p$ and $h(1) = q$. (An embedding is a continuous 1:1 function.)

Theorem 5.27. An arc-wise connected space is connected.

Exercise 5.28. Give an example of a connected space that is not arc-wise connected.

5.4 Local Connectedness

Although the closure of the topologist's comb C is connected and path connected, if we look at a small neighborhood of the point $(0, 1)$, then it's intersection with C is neither connected nor path-connected.

Definition (locally connected at a point). A space X is *locally connected at the point* p of X if and only if for each open set U containing p, there is a connected open set V such that $p \in V \subseteq U$. A space X is *locally connected* if and only if it is locally connected at each point.

Theorem 5.29. The following are equivalent:

1. X is locally connected.

2. X has a basis of connected open sets.

3. For each $x \in X$ and open set U with $x \in U$, the component of x in U is open.

4. For each $x \in X$ and open set U with $x \in U$, there is a connected set C so that $x \in \text{Int } C \subset C \subseteq U$.

5. For each $x \in X$ and open set U with $x \in U$, there is an open set V containing x and $V \subset$ (the component of x in U).

Theorem 5.30. Let X be a locally connected space and $f : X \to Y$ be an onto, closed or open map. Then Y is locally connected.

Definition (Peano Continuum). A *Peano Continuum* is a locally connected metric continuum.

Theorem 5.31. A Hausdorff space X is a Peano Continuum if and only if X is the image of $[0, 1]$ under a continuous function.

Definition (locally path connected at a point). Let X be a topological space.

1. X is *locally path* or *arc-wise connected at* p if and only if for each open set U containing p there is an open set V containing p such that for each pair of points $x, y \in V$, there is an arc in U that contains x and y. (Note: "an arc" means the homeomorphic image of $[0, 1]$).

2. A space is *locally path connected* or *locally arc-wise connected* if and only if it is locally arc-wise connected at each point.

Theorem 5.32. A locally arc-wise connected space is locally connected.

Theorem 5.33. A Peano Continuum is arc-wise connected and locally arc-wise connected.

Theorem 5.34. An open, connected subset of a Peano Continuum is arc-wise connected.

Appendix A

The real numbers

Example 17. The real numbers with the relation $\leq$ form a poset.

Definition (upper bound). Let X be a set a totally ordered set, and let $A \subseteq X$. For any $x \in X$, x is an *upper bound* of A if $y \leq x$ for all $y \in X$. If A has an upper bound we say A is said to be *bounded above*

Note that if y is an upper bound of set A, then y may or may not be an element of A.

Definition (supremum or least upper bound). The *supremum* or *least upper bound* of A, $\sup(A)$ or $\operatorname{lub} A$ is the least such upper bound; that is, $\sup(A)$ is an upper bound of A, and for any x is an upper bound of A, then $\sup(A) \leq x$.

Note that it is not necessarily the case that $\sup(S) \in S$. Note also that a set may not have an upper bound at all.

Definition (Least Upper Bound Property). Let X is a set with a total order $<$. Then X has the *least upper bound property* if any non-empty set which is bounded above has a least upper bound.

We will take as an axiom the following:

Axiom 1 (Least Upper Bound Axiom). The real numbers have the least upper bound property.

Notice that any subset of a well-ordered set is well-ordered by the same ordering restricted to the subset.

Exercise A.1. Give examples of posets that are

1. partially ordered but not totally ordered
2. totally ordered but not well ordered
3. well ordered.

Appendix B

Review of Set Theory and Logic

B.1 Set Theory

Definition (complement).

Definition (union).

Definition (intersection).

Theorem B.1 (DeMorgan's Laws).

Theorem B.2 (Generalized DeMorgan's Laws).

B.2 Logic

Michael Starbird
Department of Mathematics
RLM 8.100
The University of Texas at Austin
Austin, TX 78712
starbird@math.utexas.edu

Cynthia Verjovsky Marcotte
Department of Mathematics
St. Edward's University
3001 South Congress Ave.
Austin, TX 78704
cynthm@admin.stedwards.edu