Differential Topology Prelim

Thursday, August 18, 2016, 2:40 – 4:15 PM

Instructions: Do all three problems. If you get some parts of a multi-part problem but not others, please indicate which parts you want graded. (Skipping a part is generally viewed more favorably than writing nonsense about it.) If you need to assume the result of one part to solve another part, say that you're doing so and we'll work out appropriate partial credit.

1. Let X and Y be manifolds (without boundary), with X compact, and Z a closed submanifold of Y. Recall that a property of a smooth map $f : X \to Y$ is said to be *stable* if, for any map f_0 satisfying that property, and any smooth homotopy $F : [0,1] \times X \to Y$, there exists an $\epsilon > 0$ such that the property holds for all f_t with $t < \epsilon$. Prove that being transversal to Z is a stable property. (Yes, this is part of the Stability Theorem from Chapter 1 of Guillemin and Pollack. We're asking you to reconstruct the proof.)

2. a) Suppose that Y is a path-connected and compact n-manifold (without boundary, but not necessarily orientable), and that X is a closed and path-connected submanifold of codimension 1. Let Z be a 1-dimensional submanifold of Y. Prove that, if X divides Y into two pieces (that is, if the complement of X in Y has exactly two path components) then $I_2(X, Z) = 0$.

b) Let $Y = RP^4$, and let X be the submanifold (diffeomorphic to RP^3) obtained by setting one of the coordinates in $\mathbb{R}^5$ equal to zero. Show that X does not divide Y into two pieces.

3. Let $X = CP^1$, and let $f : X \to X$ be given by

$$f([z_0, z_1]) = [\bar{z}_0, \bar{z}_1],$$

where the bar indicates complex conjugation and $[z_0, z_1]$ is the equivalence class of $(z_0, z_1) \in \mathbb{C}^2 - \{0\}$.

a) Show that f is a smooth map.
b) Compute the degree of f
c) Find the fixed points of f. If f a Lefschetz map?
d) Compute the Lefschetz number of f.