THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part II

Thu, Aug 17, 2023

Problem 1. Let $\Omega = [0,1)$, $\mathcal{F} = \mathcal{B}[0,1)$, and $\mathbb{P} = \lambda$, where λ denotes the Lebesgue measure on $[0,1)$. For $n \in \mathbb{N}$ and $k \in \{0,1,\dots,2^n-1\}$, we define

$$I_{k,n} = [k2^{-n}, (k+1)2^{-n}),\ \mathcal{F}_n = \sigma(I_{0,n}, I_{1,n}, \dots, I_{2^n-1,n}).$$

In words, $\mathcal{F}_n$ is generated by the n-th dyadic partition of $[0,1)$. For $x \in [0,1)$, let $k_n(x)$ be the (ordinal) number of the partition element which contains x, i.e., unique number in $\{0,1,\dots,2^n-1\}$ such that $x \in I_{k_n(x),n}$. For a function $f : [0,1) \to \mathbb{R}$ we define the process $\{X_n^f\}_{n\in\mathbb{N}_0}$ by

$$X_n^f(x) = 2^n\Big(f\big((k_n(x)+1\big)2^{-n}) - f\big(k_n(x)2^{-n}\big)\Big),\ x \in [0,1).$$

(1) Show that $\{X_n^f\}_{n\in\mathbb{N}_0}$ is a martingale.
(2) Assume that the function f is Lipschitz, i.e., that there exists $K > 0$ such that $|f(y) - f(x)| \le K|y-x|$, for all $x, y \in [0,1)$. Show that the limit $X^f = \lim_n X_n^f$ exists a.s.
(3) Show that, for f Lipschitz, X^f has the property that

$$f(y) - f(x) = \int_x^y X^f(\xi)\,d\xi,\ \text{for all } 0 \le x < y < 1.$$

(*Note:* This problem gives an alternative proof of the fact that Lipschitz functions are absolutely continuous.)

Problem 2. Let $\{(B_t^1, B_t^2)\}_{t\in[0,T]}$ be a two-dimensional Brownian motion. Given $(a_1, a_2, b) \in \mathbb{R}^3$ find the distribution of the random time T given by

$$T = \inf\{t \ge 0 \,:\, a_1 B_t^1 + a_2 B_t^2 = b\}$$

(*Note:* Any of the following will be accepted: pdf, cdf, Laplace transform, characteristic function.)

Problem 3. Let $\{B_t\}_{t\in[0,1]}$ be a Brownian motion on $[0,1]$ and let $\{H_t\}_{t\in[0,1]}$ be a *deterministic* process with $\int_0^1 H_u^2\,du < \infty$. Show that the random variable $\int_0^1 H_u\,dB_u$ is normally distributed.

For extra credit give an example of a non-deterministic, progressive, and B-integrable process H such that $\int_0^1 H_u\,dB_u$ is normal.