PRELIMINARY EXAMINATION IN TOPOLOGY: PART 2

January 2023 2 Hours

Work all 3 problems. Explain your work carefully. The problems are weighted equally.

1. State clearly whether each of the following unrelated assertions is true or false. *Prove* your answer with a deductive argument, an example, a counterexample, etc.

   (a) Let Σ be a smooth connected oriented 2-manifold with Euler number -2. Then there exists a degree 1 smooth map $S^1 \times \Sigma \to S^3$.

   (b) Let $L \subset \mathbb{A}^3$ be an affine line in real affine 3-space. If $f\colon \mathbb{R} \to \mathbb{A}^3$ is transverse to L, then $f(\mathbb{R}) \cap L = \emptyset$.

   (c) Suppose $f\colon \mathbb{RP}^3 \to \mathbb{RP}^3$ is smooth. Then the Lefschetz number $L(f)$ is odd.

2. Let C denote the subset of the real affine plane $\mathbb{A}^2$ with coordinates x, y that is cut out by the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

   (a) Draw a picture of C and display the orientation as the boundary of the compact region C bounds.

   (b) Compute

$$\int_C x\,dy.$$

   Use the orientation of C in part (a).

3. Let Y be a smooth manifold and suppose $Z \subset Y$ is a closed submanifold. For each of the following pairs $Z \subset Y$, either (i) produce a compact manifold X and a smooth map $f\colon X \to Y$ such that the mod 2 intersection number $\#_2(f, Z) = 1$, or (ii) prove that no such X and f exist.

   (a) $S^1 \subset S^3$, where S^1 is equatorial

   (b) $\mathbb{RP}^1 \subset \mathbb{RP}^3$, the image of (a) under the quotient by the antipodal map

   (c) $S^2 \times \mathrm{pt} \subset S^2 \times S^2$

   (d) $\mathrm{pt} \times \mathbb{RP}^2 \subset \mathbb{RP}^2 \times \mathbb{RP}^3$, where $\mathbb{RP}^2 \subset \mathbb{RP}^3$ is the projectivization of a subspace $\mathbb{R}^3 \subset \mathbb{R}^4$