Differential Topology Prelim Exam
August 15, 2023, 15:00–17:00

Solve all three problems.

Problem 1. True or false with brief explanations, including counterexamples when false.

(a) Suppose M is a manifold and $f : M \to \mathbb{R}$ is smooth. Then f is a Morse function if and only if every one of its critical points is isolated.

(b) Every smooth map $\mathbb{C}P^4 \to S^8$ is nullhomotopic.

(c) Every smooth map $\mathbb{C}P^4 \to S^9$ is nullhomotopic.

Problem 2. Let X be the subset $x^n - y^2 = 0$ of $\mathbb{C}^2$, where $n > 2$ is even. Prove that X is the union of two smooth submanifolds of $\mathbb{C}^2$, that meet each other non-transversely. Describe the natural orientations on these submanifolds, and on $\mathbb{C}^2$, and find the local intersection number at their point of intersection.

Problem 3. Write T^k for the k-torus $\mathbb{R}^k/\mathbb{Z}^k$. On the space of 1-forms $\Omega^1(T^k)$, define an averaging operator $A : \omega \mapsto \int_{t \in T^k} L_t^*(\omega) d\mu$, where L_t means translation by t and $d\mu$ is Lebesgue measure. Prove that A defines an isomorphism from $H^1_{dR}(T^k)$ to the cotangent space $T_0^*(T^k)$.